Björn S. Rüffer

Monotone dynamical systems, graphs, and stability of large-scale interconnected systems

Monotone dynamical systems, graphs, and stability of large-scale interconnected systems

von Björn Sebastian Rüffer

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Prof. Dr. Lars Grüne (Universität Bayreuth)

To my parents, Antje and Herbert.

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Introduction

This work is devoted to the quest for sufficient criteria for the stability of large scale nonlinearly interconnected control systems such that these criteria can be stated in a 'closed' form.

We have been inspired by the linear case: Given n linear one dimensional control systems

$$\dot{x}_i = -a_i x_i + \sum_{j=1}^{n} b_{ij} x_j + d_i u_i, \quad i = 1, \ldots, n,$$

where $x_i, a_i, b_{ij}, d_i \in \mathbb{R}$ for $i, j = 1, \ldots, n$ such that $a_i > 0$ for all $i = 1, \ldots, n$. The state variables x_i of the individual systems are one dimensional, so we can write the n single equations in vector notation to obtain

$$\dot{x} = \begin{pmatrix} -a_1 & b_{12} & b_{13} & \ldots & b_{1n} \\ b_{21} & -a_2 & b_{23} & \ldots & b_{2n} \\ & & \vdots & & \\ b_{1n} & \ldots & \ldots & b_{n,n-1} & -a_n \end{pmatrix} x + Du, \qquad (\dagger)$$

where we write $x = (x_1, \ldots, x_n)^\top$, $u = (u_1, \ldots, u_n)^\top$, and $D = \operatorname{diag}(d_1, \ldots, d_n)$. Denoting the matrix in $(\dagger)$ by A, then it is well known that the origin is globally asymptotically stable for the compound system $(\dagger)$ with inputs u set to zero, if and only if A is a Hurwitz matrix, i.e., all eigenvalues of A lie in the left open complex half plane.

Observe that this is a matrix stability criterion. In this case the matrix A encodes all information that we have about the systems, excluding information about external inputs u. Also this criterion is not only sufficient but also necessary.

If the interconnected systems themselves are nonlinear and the interconnection structure, i.e., how individual systems influence each other, is also nonlinear, then of course we cannot directly obtain a matrix to investigate from the individual systems themselves. Still, under some circumstances stability properties of equilibriums of the compound nonlinear system might be accessible through different methods. Either we could try to prove stability

properties for the large-scale system directly, or otherwise we could decompose the compound system into smaller components, assess these individually and somehow aggregate the obtained results to establish stability of the compound system. Such a divide and conquer method could be, e.g, to assess simple interconnections like cascades and simple feedback loops between the individual systems and so, in an iterative manner, aggregate several stable systems to one stable system in each step, until only one stable system is left.

Of course, by applying such a method we are able to prove stability of the compound system for certain classes of systems and stability concepts. At this stage the concept of input-to-state stability comes into the game. This is a nonlinear concept for robust stability. For this stability concept it is known, that cascades of stable systems and, under a small gain condition, feedback loops of stable systems are stable.

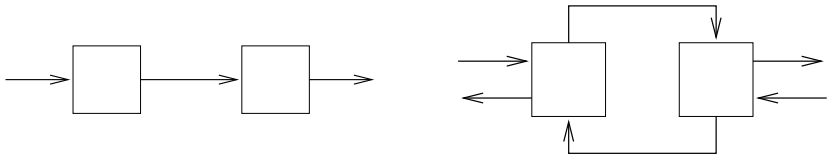

Figure 0.1.: A cascade and a feedback interconnection of two input-output systems.

Now we ask for a criterion similar to the one in the linear case, roughly stating something similar to "A is Hurwitz, so the network must be stable", but for general interconnection types.

Input-to-state stability in essence gives us a matrix of nonlinear functions that describe worst case bounds on the interactions of the interconnected systems. We state and prove conditions on that matrix that ensure stability of the network. In order to do so, we need something equivalent to matrix theory, but for 'nonlinear' matrices. Therefore the content of this work comprises two subjects. The first one deals with monotone operators on the positive orthant in $\mathbb{R}^n$. We may think of these operators as the matrices just mentioned. The second subject is a stability analysis of large-scale interconnections of dynamical systems, where we heavily depend on the "nonlinear matrix theory", i.e., the theory of monotone operators that we develop first.

Monotone Operators

The first subject to be studied concerns monotone operators $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = (\mathbb{R}_+)^n$, $\mathbb{R}_+ = [0, \infty)$. We investigate a condition that implies for a class of such operators that

(i) the origin is globally asymptotically stable for the dynamical system

$$x_{k+1} = T(x_k); \qquad (*)$$

(ii) for $b \in \mathbb{R}^n_+$ the inequality

$$x \leq T(x) + b$$

possesses a nonnegative solution x; and all solutions x satisfy $\|x\| \leq \varphi(\|b\|)$ for a continuous and monotone function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\varphi(0) = 0$;

(iii) there exists a function $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ that is strictly increasing and unbounded in every component and satisfies

$$T(\sigma(t))_i < \sigma(t)_i$$

for all components $i = 1, \ldots, n$ and all $t > 0$.

The condition of interest is essentially that for each $x \neq 0$, there exists a component i, such that

$$T(x)_i < x_i.$$

This condition will be called "no joint increase condition", because there always (except at the origin) has to be one decreasing component.

In the literature on monotone dynamical systems [HS05, Smi95] more interest is put into the questions, whether there exist fixed points or periodic orbits of T, i.e., equilibrium states of the system $(*)$, and whether or not these are stable, attractive and so on. Property (iii) is related to the Perron-Frobenius Theorem for nonnegative matrices, which has nonlinear generalizations, see, e.g., [GG04] and the references therein.

If T is a linear or max-linear operator, then T can be represented by a nonnegative matrix. It is known that in this case properties (i) and (ii) are true if the spectral radius of T is less than one. If T is in addition irreducible, then also (iii) holds [Lur05, LT85, BP79]. A nonlinear generalization in the max-case is given in [Pot02, Tee05].

In the general nonlinear case far less is known and in general the no joint increase condition does not imply anything from our list, cf. Example 2.1.6.

We show in Lemma 2.0.1 that (iii) also holds for general nonnegative matrices. The max-linear case as well as the corresponding results in [Pot02, Tee05] are included in our Propositions 2.3.16 and 2.3.17.

If we restrict our attention to a class of matrix-like operators then some of the ideas from the linear theory carry over to the nonlinear setting. To

this end let $\Gamma = (\gamma_{ij})_{i,j=1}^n$ be a matrix whose entries are continuous, strictly increasing, and unbounded functions $\gamma_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$. Such a matrix defines a monotone operator in various ways, e.g., for $s \in \mathbb{R}_+^n$ we define

(a) $T(s) = \Gamma_\oplus(s)$ by $\Gamma_\oplus(s)_i = \max_j \gamma_{ij}(s_j)$, for $i = 1, \dots, n$;

(b) $T(s) = \Gamma_\Sigma(s)$ by $\Gamma_\Sigma(s)_i = \sum_j \gamma_{ij}(s_j)$, for $i = 1, \dots, n$;

(c) $T(s) = \Gamma_{\|.\|}(s)$ by $\Gamma_{\|.\|}(s)_i = \|(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n))^\top\|$, for $i = 1, \dots, n$, for a monotone norm $\|.\|$ on $\mathbb{R}^n$.

The first case has already been studied in [Tee05, PAQ03, Pot02] and it turns out that this case has the strongest analogy to the linear case, see Section 2. There we will show that (i)–(iii) and the no joint increase condition are all equivalent for the operator defined in (a), just as in the linear and in the max-linear case.

For the cases (b) and (c) and even more general cases we will show that if there exists a monotone diagonal operator D, such that the operator $\tilde{T} := T \circ D$ satisfies the no joint increase condition, then we will have (i)–(iii) and in some sense also vice versa, see Theorem 2.2.9, Theorem 2.5.1, and Remark 2.5.2

The essential arguments will be of graph theoretical nature: A nonnegative $n \times n$-matrix A gives rise to a graph with vertices $1, \dots, n$: This graph has an edge directed from j to i whenever $a_{ij} > 0$. When we restrict our attention to connected graphs, the important observation is, that graphs can be divided in the three classes R, I, and P.

The first class R consists of graphs that can be drawn as a cascade, i.e, they have a tree like structure. The class I consists of graphs in which there exists a path from every vertex to any other vertex. If these paths can in addition to be chosen to be of same length independent of the vertices they connect, then the graph is of class P. Clearly we have $P \subset I$. Translated back to matrices we have reducible matrices (R), irreducible (I), and primitive (P) matrices. If we consider the dynamical system ($*$) then irreducibility of the operator T tells us, that the dynamics of the ith component is eventually affected by the dynamics of the jth component, for any j. This for example implies, that not only one component of a trajectory of ($*$) can be unbounded.

Another important concept is that of decay sets. These are sets where the image points are in every component less than the original points, e.g.,

$$\Omega = \{x \in \mathbb{R}_+^n : T(x)_i < x_i \text{ for all } i = 1, \dots, n\}$$

is such a decay set. In Section 1.5 it is shown, that these sets are always unbounded and pathwise connected, if the no joint increase condition holds. If

T is in addition irreducible, then these sets are unbounded in every direction, see for example Theorem 2.2.1.

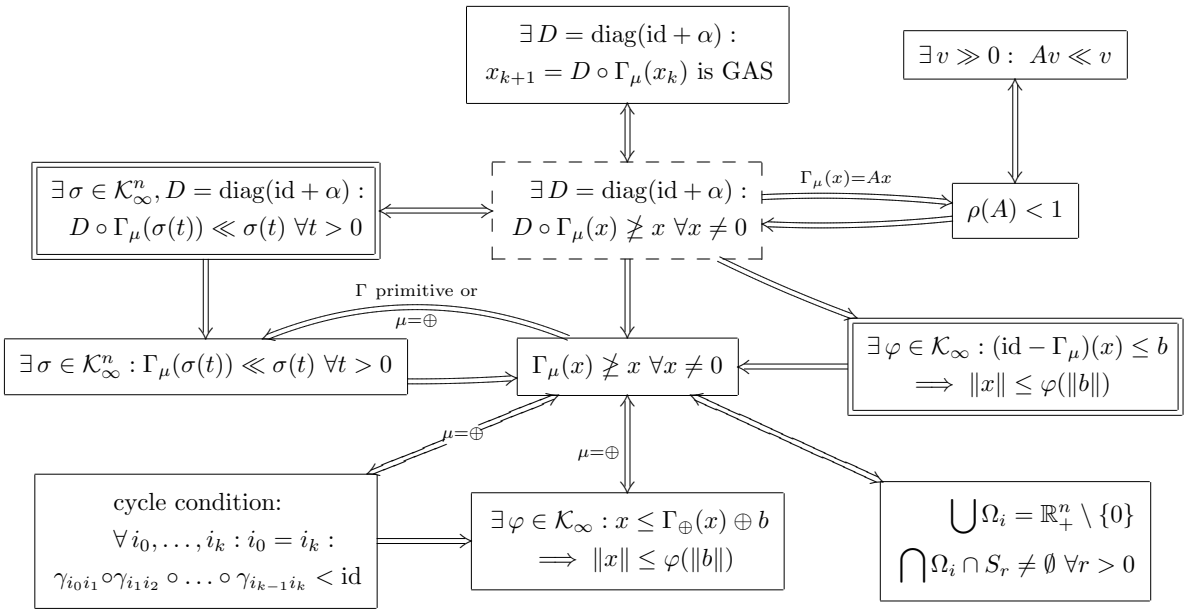

Figure 0.2.: The implication diagram of Chapter 2. Here $\Gamma \in \mathcal{G}^{n\times n}$, $\mu \in \mathrm{MAF}_n^n$, $\alpha \in \mathcal{K}_\infty$, $A \in \mathbb{R}_+^{n\times n}$, $x, b, v \in \mathbb{R}_+^n$ and $\rho(A)$ denotes the spectral radius of the matrix A. The sets Ω_i are the sets where Γ_μ decreases in the ith component, the set S_r is the sphere of radius $r > 0$ with respect to the 1-norm.

Figure 0.2 indicates the relations that we will prove in Chapter 2. The dashed box is the no joint decrease condition in its 'robust' form. From here we can get to every other property in the implication graph. In the double frames the two most important implications for Chapter 3 are stated.

Large scale interconnections of dynamical systems

In Chapter 3 we are interested in the preservation of stability properties of control systems under general interconnections. It may seem that the second subject of this thesis is not connected to the first subject, but it is. In fact, the first subject grew out of the second. The results on monotone operators of Chapter 2 are essential for this chapter.

A control system can either be of continuous form and is usually given as

by differential equation

$$\dot{x} = f(x, u), \quad y = h(x), \tag{Σ}$$

where $x \in \mathbb{R}^N$ is the state, $y \in \mathbb{R}^P$ is the output, and $u \in \mathbb{R}^M$ is the input of the system.

There are several concepts of stability. We investigate a few such concepts that are preserved under cascades and feedback loops, possibly if additional assumptions are made. This means, we consider two systems

$$\dot{x}_1 = f_1(x_1, u_1), \quad y_1 = h_1(x_1) \tag{0.1}$$
$$\dot{x}_2 = f_2(x_2, u_2), \quad y_2 = h_2(x_2) \tag{0.2}$$

and put them into cascade interconnection by $u_1 = g_{12}(y_2)$ or in feedback interconnection by

$$u_1 = g_{12}(y_2) \text{ and } u_2 = g_{21}(y_1).$$

The questions is under which condition the resulting system is again stable. These two interconnection structures have been considered, e.g., in [CPT95, Grü02b, JMW96, JTP94, Son89] in general nonlinear settings.

Moreover, we are concerned with the preservation of stability properties for more general interconnection structures involving a large number of individual systems.

To this end we consider n interconnected control systems given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, y_2 \ldots, y_n, u_1), \quad y_1 = h_1(x_1), \\ &\vdots \\ \dot{x}_n &= f_n(y_1, \ldots, y_n, x_n, u_n), \quad y_n = h_n(x_n). \end{aligned}$$

Such large scale interconnections have also been studied in [Bai65, MM77, RHL77, Šil79, Vid81].

In [MM77] sufficient conditions for the asymptotic stability of a composite system are stated in terms of the negative definiteness of some test matrix, while in [RHL77] conditions for the stability of interconnected systems in terms of Lyapunov functions of the individual systems are obtained.

In [Šil79] structural perturbations and their effects on the stability of composite systems are considered using Lyapunov theory. The method is to reduce each subsystem to a one-dimensional one, such that the stability properties of the reduced representation imply the same stability properties of the original interconnected system. In some cases the lower dimensional representation gives rise to an interconnection matrix $\bar{W}$, such that quasi

dominance or negative definiteness of $\bar{W}$ yield asymptotic stability of the composite system.

In [Vid81] small gain type theorems for general interconnected systems with linear gains can be found. These results are of the form that the spectral radius of a gain matrix should be less than one to conclude stability. The result obtained here may be regarded as a nonlinear generalization in the same spirit.

In [Bai65] linear interconnections are investigated and a linear small-gain condition is established for simple closed loops. Moreover, arbitrary linear interconnections are asymptotically stable in the large (ASL), if an auxiliary continuous time linear system is ASL. We provide nonlinear results of the same spirit in Chapter 3. The stability concepts we consider can be formulated using comparison functions and derive from input-to-state stability, a concept first introduced in [Son89].

For example, we could assume that each system is input-to-state stable and that we have $y_i = x_i$ for all i. That is, for each $i = 1, \dots, n$ there holds an estimate on the solutions

$$\|x_i(t)\| \leq \beta_i(\|x_i(0)\|, t) + \sum_{j=1,\ j\neq i}^{n} \gamma_{ij}(\|x_{j\,[0,t]}\|_\infty) + \gamma_i(\|u_i\|_\infty) \tag{0.3}$$

for all $t \geq 0$.

The term $\beta_i(\|x_i(0)\|, t)$ is increasing in the first argument and tends to zero in the second. This term captures the transient behavior of the individual system. The functions $\gamma_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are either zero or strictly increasing, continuous, and unbounded. These functions are called gains. They measure, how much one system may affect the other.

We may now write the gains in matrix form $\Gamma = (\gamma_{ij})$. In Chapter 3 it turns out, that if Γ_Σ defined by (b) satisfies the no joint increase condition in a robust sense, i.e., there exists a special monotone operator D, such that $\tilde{T} := \Gamma_\Sigma \circ D$ satisfies the no joint increase condition, then the interconnected system

$$\dot{x} = f(x, u) \tag{$**$}$$

with $x = (x_1^\top, \dots, x_n^\top)^\top$, $u = (u_1^\top, \dots, u_n^\top)^\top$, and $f = (f_1, \dots, f_n)^\top$ is also input-to-state stable. In Theorem 3.3.4 we prove this also for arbitrary 'aggregations' of the gains in (0.3) instead of a sum.

We will see that this stability condition is equivalent to the existence of an operator $\tilde{D}$, such that the discrete system $(*)$ with $T = \Gamma_\Sigma \circ \tilde{D}$ is globally asymptotically stable. We also consider Lyapunov type stability notions and construct a Lyapunov function for $(**)$, see Theorem 3.3.12.

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1. Preliminaries

1.1. Comparison functions

In control theory important generalizations of linear theory to nonlinear settings rely on the concept of comparison functions. Here we use them in a rather abstract context.

Let $\mathbb{R}_+$ denote the nonnegative real numbers. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if γ is strictly increasing, continuous, and satisfies $\gamma(0) = 0$. The function γ is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. By $\mathcal{G}$ we denote the set of functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ that are either of class $\mathcal{K}_\infty$ or constantly zero. Class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions are commonly referred to as comparison functions, because of their use as a nonlinear replacement for the ratio between quantities. For comparison functions α, β we write $\alpha < \beta$ if $\alpha(t) < \beta(t)$ for all $t > 0$.

Lemma 1.1.1 *Let $\rho \in \mathcal{K}_\infty$, then its inverse $\rho^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ exists and is of class $\mathcal{K}_\infty$.*

Proof. Clearly the image of ρ is $\mathbb{R}_+$, hence ρ is onto; the function ρ is strictly increasing, so it is one-to-one. Therefore ρ^{-1} exists, is strictly increasing, satisfies $\rho^{-1}(0) = 0$, and ρ^{-1} is unbounded. Since ρ maps open intervals to open intervals, ρ^{-1} is continuous. □

Note that class $\mathcal{K}$ has a semigroup structure with respect to composition and has the identity as neutral element. Similarly, class $\mathcal{K}_\infty$ has a group structure.

A function $\beta : \mathbb{R}_+^2 \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for every fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$, while for every fixed $s \geq 0$ the function $\beta(s, \cdot)$ is nonincreasing and tends to zero as the argument approaches infinity. We call a $\mathcal{KL}$ function β *positive* if β is positive definite with respect to the first argument, i.e., for $s > 0$ and $t \geq 0$, $\beta(s, t) > 0$.

An important generalization of the triangular inequality for norms is the weak triangular inequality. Here we state a slight variation of a formulation (see (1.1)) given in [JTP94]:

Lemma 1.1.2 *Let $\gamma \in \mathcal{K}$ and let $\rho \in \mathcal{K}_\infty$. Then for all $a, b \in \mathbb{R}_+$ we have*

$$\gamma(a+b) \leq \gamma \circ (id+\rho)(a) + \gamma \circ (id+\rho^{-1})(b).$$

Proof. In [JTP94] it is shown that for any $\alpha \in \mathcal{K}_\infty$, such that $\alpha - \mathrm{id} \in \mathcal{K}_\infty$, the inequality

$$\gamma(a+b) \leq \gamma \circ \alpha(a) + \gamma \circ \alpha \circ (\alpha - \mathrm{id})^{-1}(b) \tag{1.1}$$

holds for all $a, b \geq 0$. For $\rho \in \mathcal{K}_\infty$ define $\alpha := \mathrm{id} + \rho$ and by (1.1) the claim follows. □

Another result in this direction is the following, stating that functions of the form $(\mathrm{id} + \rho)$, $\rho \in \mathcal{K}$, can be factorized into similar terms.

Lemma 1.1.3 *1. For any $\rho \in \mathcal{K}$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}$, such that*

$$\rho + id = (\alpha_1 + id) \circ (\alpha_2 + id).$$

Moreover, if $\rho \in \mathcal{K}_\infty$, then also α_i, $i = 1, 2$, can be chosen to be of class $\mathcal{K}_\infty$.

2. For any $\rho \in \mathcal{K}_\infty$, $\rho > id$, and any $k \geq 0$, there exists $\rho_1^{(k)}, \rho_2^{(k)} \in \mathcal{K}_\infty$, such that $\rho_1^{(k)} > id$, $\rho_2^{(k)} > id$, and

$$\rho = \rho_1^{(k)} \circ \rho_2^{(k)},$$

in such a way that for all $k \geq 0$ we have $\rho_2^{(k+1)}(t) > \rho_1^{(k)}(t)$ for all $t > 0$.

Proof. 1. Note that $(\frac{1}{2}\rho + id)$ is of class $\mathcal{K}_\infty$. Now choose, e.g., $\alpha_2 = \frac{1}{2}\rho$ and $\alpha_1 = \frac{1}{2}\rho \circ (\frac{1}{2}\rho + id)^{-1}$. Then

$$(\alpha_1 + \mathrm{id}) \circ (\alpha_2 + \mathrm{id}) = \alpha_1(\alpha_2 + \mathrm{id}) + \alpha_2 + \mathrm{id} =$$
$$\frac{1}{2}\rho \circ (\frac{1}{2}\rho + id)^{-1} \circ (\frac{1}{2}\rho + \mathrm{id}) + \frac{1}{2}\rho + id = \frac{1}{2}\rho \circ \mathrm{id} + \frac{1}{2}\rho + \mathrm{id} = \rho + \mathrm{id}.$$

2. For $\lambda^{(k)} \in (0,1)$ let $\rho_1^{(k)} := \lambda^{(k)}\rho + (1 - \lambda^{(k)})\mathrm{id}$. Clearly $\rho_1^{(k)} \in \mathcal{K}_\infty$ and $\mathrm{id} < \rho_1^{(k)} < \rho$. This implies that $\rho_2^{(k)} := (\rho_1^{(k)})^{-1} \circ \rho > \mathrm{id}$. As the composition of $\mathcal{K}_\infty$ functions, $\rho_2^{(k)}$ is also of class $\mathcal{K}_\infty$.

Clearly $\lambda^{(k+1)} < \lambda^{(k)}$ implies $\rho_1^{(k+1)}(t) < \rho_1^{(k)}(t)$ and $\rho_2^{(k+1)}(t) > \rho_2^{(k)}(t)$ for $t > 0$.

Now choose, e.g., $\lambda^{(k)} := \frac{1}{k+2}$ to obtain a strictly decreasing sequence $\{\lambda^{(k)}\}_{k \geq 0}$ satisfying $\lambda^{(k)} \in (0, \frac{1}{2}]$ for all $k \geq 0$. □

This result extends easily to the following type of diagonal operators, that will be of great importance in Section 2.2:

By $\mathbb{R}_+^n$ we denote $(\mathbb{R}_+)^n = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$. Let $\alpha \in \mathcal{K}_\infty$ and define $D : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by $D(s) = (\alpha(s_1), \ldots, \alpha(s_n))^\top$ for $s \in \mathbb{R}_+^n$. For short we write $D = \mathrm{diag}(\alpha)$.

Lemma 1.1.4 *1. Let $D = \mathrm{diag}(id + \alpha)$ for some $\alpha \in \mathcal{K}_\infty$. Then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that for $D_i = \mathrm{diag}(id + \alpha_i)$, $i = 1, 2$,*

$$D = D_1 \circ D_2.$$

2. Let $D = \mathrm{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty$ such that $\rho > id$. Then for any $k \geq 0$ there exist $\rho_1^{(k)}, \rho_2^{(k)} \in \mathcal{K}_\infty$ satisfying $\rho_i^{(k)} > id$, such that for $D_i^{(k)} = \mathrm{diag}(\rho_i^{(k)})$, $i = 1, 2$,

$$D = D_1^{(k)} \circ D_2^{(k)}.$$

Moreover, $D_2^{(k)}$, $k \geq 0$, can be chosen such that for all $0 \ll s$ we have

$$D_2^{(k)}(s) \ll D_2^{(k+1)}(s),$$

where $\ll$ means componentwise strictly less, see Subsection 1.2.1.

Proof. This is a componentwise application of Lemma 1.1.3. □

Lemma 1.1.5 *Let $\rho \in \mathcal{K}_\infty$. Then there exists a $\tilde{\rho} \in \mathcal{K}_\infty$ such that $(id + \rho)^{-1} = id - \tilde{\rho}$.*

Proof. Just define $\tilde{\rho} = \rho \circ (\mathrm{id} + \rho)^{-1}$. Then $(\mathrm{id} - \tilde{\rho}) \circ (\mathrm{id} + \rho) = (\mathrm{id} + \rho) - \tilde{\rho} \circ (\mathrm{id} + \rho) = \mathrm{id} + \rho - \rho \circ (\mathrm{id} + \rho)^{-1} \circ (id + \rho) = \mathrm{id} + \rho - \rho = \mathrm{id}$, which proves the lemma. □

Class $\mathcal{K}$ functions are not necessarily smooth. Nevertheless, locally they can always be approximated arbitrarily close by smooth $\mathcal{K}$-functions.

Lemma 1.1.6 *Let $\rho \in \mathcal{K}$. Then for any $\varepsilon > 0$ and $M > 0$ there exists a $\tilde{\rho} \in \mathcal{K} \cap \mathcal{C}^\infty([0, \infty[)$, such that $\sup_{x \in [0,M]} |\rho(x) - \tilde{\rho}(x)| < \varepsilon$.*

Here $\mathcal{C}^\infty([0, \infty[)$ is the set of infinitely often differentiable functions for which the one sided limits of the difference quotient at the origin exist.

Proof. Using standard smoothing techniques from functional analysis, see, e.g., [Eva98, pp.629ff], this fact is well known. The technical steps are as

follows: For notational simplicity we identify with ρ its extension $\bar{\rho}$ to all of $\mathbb{R}$ defined by

$$\bar{\rho}(x) = \begin{cases} \rho(x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Given $\delta > 0$ denote

$$\eta_\delta(x) := \begin{cases} C \cdot 1/\delta \cdot \exp\left(\frac{1}{(x^2/\delta^2)-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where C is chosen so that $\int_{\mathbb{R}} \eta_\delta(x)dx = 1$. Note that this constant is independent of the choice of $\delta > 0$. We have $\eta_\delta \in \mathcal{C}^\infty(\mathbb{R})$. Then the mollification of ρ is defined by the convolution

$$\rho^\delta(x) := (\eta_\delta * \rho)(x) := \int_{\mathbb{R}} \eta_\delta(x-y)\rho(y)dy = \int_{-\delta}^{\delta} \eta_\delta(y)\rho(x-y)dy.$$

The function ρ^δ is strictly increasing, since for all $x, t > 0$ we have

$$\rho^\delta(x+t) = \int_{-\delta}^{\delta} \eta_\delta(y)\rho(x+t-y)dy > \int_{-\delta}^{\delta} \eta_\delta(y)\rho(x-y)dy = \rho^\delta(x),$$

since ρ is strictly increasing. Therefore the functions $\tilde{\rho}^\delta$ defined by $\tilde{\rho}^\delta(t) := \rho^\delta(t) - \rho^\delta(0)$ are of class $\mathcal{K}$.

Now by [Eva98, Theorem C.6, p.630], ρ^δ is in $\mathcal{C}^\infty(\mathbb{R})$ and converges to ρ uniformly on compact sets as $\delta \to 0$, hence so does $\tilde{\rho}^\delta$. This implies the result. □

1.2. Order and monotonicity

Let $\mathbb{R}$ denote the real numbers and $\mathbb{R}_+$ the nonnegative real numbers. Let X be a real vector space. For two sets $A, B \subset X$ we define $A + B = \{a + b : a \in A, b \in B\}$, $-A = \{-a : a \in A\}$, and $\mathbb{R}_+ \cdot A = \{r \cdot a : a \in A, r \in \mathbb{R}_+\}$. The set $\mathrm{co}(A)$ consists of all convex combinations of elements of A. Given a topology, by $\operatorname{int} A$ we denote the interior of A and by $\overline{A}$ its closure.

1.2.1. Order

In this work all vector spaces are real vector spaces. A *partial order* on a subset X of a Banach space is a relation $\leq$ that is

1. reflexive: $x \leq x$ for all $x \in X$;

2. transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$;

3. antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$.

We write $x < y$ if $x \leq y$ and $x \neq y$. We assume that the order relation is closed, i.e., compatible with the topology on X, that is $x \leq y$ if $x_k \to x$, $y_k \to y$ and $x_k \leq y_k$ for all k.

Most of the time we will be interested only in the *positive cone* X_+ of X, that is a nonempty closed subset of X with the properties

1. $\mathbb{R}_+ \cdot X_+ \subset X_+$,

2. $X_+ + X_+ \subset X_+$,

3. $X_+ \cap (-X_+) = \{0\}$.

In the future by cone we always refer to a positive cone. When a cone is given, the relation

$$x \leq y \text{ if and only if } y - x \in X_+$$

is a closed partial order relation on X. We say that the partial order $\leq$ is generated by the cone X_+.

If a partial order relation is generated by a cone X_+ and X_+ has nonempty interior $\operatorname{int} X_+$, then we write $x \ll y$ whenever $y - x \in \operatorname{int} X_+$.

For $A, B \subset X$ we write $A \leq B$ ($A < B$) if for all $a \in A$ and all $b \in B$ we have $a \leq b$ ($a < b$). The set A is *majorized* (*minorized*), if there exists $a_0 \in X$, such that $a \leq a_0$ ($a \geq a_0$) for all $a \in A$. Then a_0 is called a *majorant* or *upper bound* (*minorant* or *lower bound*) of A. If A is both majorized and minorized, then it is *order bounded*. For given $a \leq b \in X$ the set $[a, b] := \{s \in X : a \leq s \leq b\}$ is an *order interval*. We will also use the notation $]a, b[:= \{s \in X : a \ll s \ll b\}$. Other order interval as $[a, b[$, $]a, b]$ are defined in the obvious manner. If there exists a majorant a_0 of set A such that $a_0 \leq b$ for any other majorant b of A, then $a_0 =: \sup A$ is called the supremum of A. The infimum $\inf A$ is defined analogously.

Note that the supremum and infimum in general, even for finite sets, need not to exist:

Example 1.2.1 *Let $X = \mathbb{R}^2$ and $X_+ = \mathbb{R}_+ \cdot (1, 0)^\top$, i.e., the positive cone is spanned by the first unit vector. Consider the partial order generated by this cone. This partial order is closed. We have $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 = y_2$.*

Hence points not lying on the same line parallel to the axis spanned the first unit vector are not comparable. For example, there is no vector being simultaneously greater than $(0,1)^\top$ *and* $(0,2)^\top$*. So* $\sup\{(0,1)^\top,(0,2)^\top\}$ *is not defined.*

A *lattice* is a partially ordered set $(X,\leq)$ where the two binary operations defined by

1. $a \vee b := \sup\{a,b\}$ and
2. $a \wedge b := \inf\{a,b\}$

exist for all $a,b \in X$.

A Banach space with closed partial order that is a lattice will be called a *vector lattice*.

Example 1.2.2 *The cone* $\mathbb{R}^n_+$ *of vectors in* $\mathbb{R}^n$ *with nonnegative components generates the following order relations:*

1. $x \leq y$ *if* $x_i \leq y_i$ *for* $i = 1,\ldots,n$,
2. $x < y$ *if* $x_i \leq y_i$ *for* $i = 1,\ldots,n$, *and* $x_i < y_i$ *for at least one* i,
3. $x \ll y$ *if* $x_i < y_i$ *for* $i = 1,\ldots,n$.

With this partial order relation $\mathbb{R}^n$ *is a vector lattice.*

We will also be interested in the logical negation of the order relation $\leq$, namely $x \not\leq y$ if and only if $x \leq y$ is false. Note that this is not the same as saying $x > y$, except for $n = 1$. On $\mathbb{R}^n$ we have $x \not\leq y$ if there exists an i, such that $x_i > y_i$. For the case $n = 2$ with the order induced by $\mathbb{R}^2_+$ for some given $y \in \mathbb{R}^2_+$ in Figure 1.2.1 the hatched region is the set of points $x \in \mathbb{R}^2_+$ satisfying $x \not\geq y$.

1.2.2. Monotonicity

Let X be a vector lattice with a closed partial order $\leq$ induced by a positive cone X_+ with nonempty interior. Let $T : X_+ \to X_+$. For $A \subset X_+$ by $T(A)$ we denote $\{T(a) : a \in A\}$ and for $a \in X_+$ we define $T^k(a) = T \circ \ldots \circ T(a) = T(\ldots(T(a)))$, the k-times successive application of T to a.

An operator $T : X_+ \to X_+$ is

1. *monotone*, if $T(x) \leq T(y)$ for all $x,y \in X_+$ satisfying $x \leq y$;
2. *strictly monotone*, if $T(x) < T(y)$ for all $x,y \in X_+$ satisfying $x < y$.
3. *strongly monotone*, if $T(x) \ll T(y)$ for all $x,y \in X_+$ satisfying $x < y$;

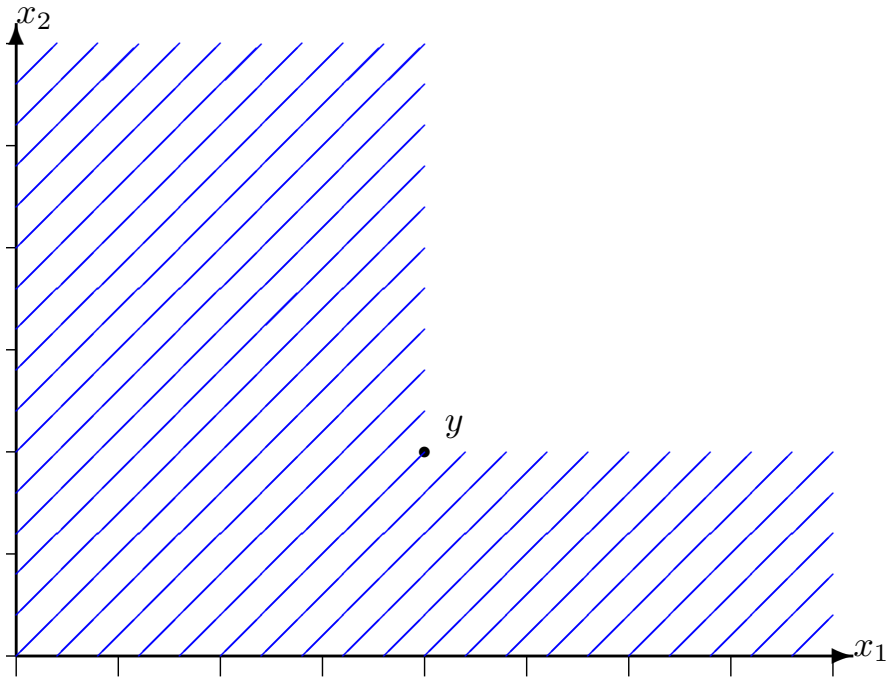


Figure 1.1.: The region $\{x \in \mathbb{R}^2_+ : x \ngeq y\}$.

4. *strictly increasing* , if $T(x) \ll T(y)$ for all $x, y \in X_+$ satisfying $x \ll y$;

5. *eventually strongly monotone*, if there exists a $k_0 > 0$ such that $T^k(x) \ll T^k(y)$ for all $k \geq k_0$ and all $x, y \in X_+$ satisfying $x < y$;

6. *strongly order-preserving* (SOP), if T is monotone, and when $x < y$ there exist respective neighborhoods U, V of x, y and $k_0 \geq 1$ such that $T^k(U) \leq T^k(V)$ for all $k \geq k_0$.

We have the following obvious implications: 3. $\Longrightarrow$ 4., 3. $\Longrightarrow$ 2. $\Longrightarrow$ 1. and 3. $\Longrightarrow$ 5.

For $x \in \mathbb{R}^n$ denote by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

the p-norm of x if $p \in [1, \infty[$ and by

$$\|x\|_\infty = \max |x_i|$$

if $p = \infty$. By the symbol $\|\cdot\|$ we refer to any fixed norm on $\mathbb{R}^n$, if no further requirements on the norm are needed.

A *monotone aggregation function* on $\mathbb{R}^n_+$ is a map $m : \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfying the following axioms:

(M1) positive definiteness: $m(x) \geq 0$ for all $x \in \mathbb{R}_+^n$ and $m(x) = 0$ if and only if $x = 0$;

(M2) strict monotonicity: $m(x) < m(y)$ if $x < y$;

(M3) unboundedness: $m(x) \to \infty$ if $\|x\| \to \infty$;

(M4) subadditivity: $m(x + y) \leq m(x) + m(y)$ for all $x, y \in \mathbb{R}_+^n$;

(M5) continuity: m is continuous.

The set of all such monotone aggregation functions on $\mathbb{R}_+^n$ is denoted by MAF_n. The easiest examples of monotone aggregation function are monotone norms on $\mathbb{R}^n$, i.e., norms $\|\cdot\|_M$ that satisfy $\|x\|_M \leq \|y\|_M$ if $|x_i| \leq |y_i|$ for $i = 1, \ldots, n$. These include the Euclidean norm, the 1-norm, and the max-norm. More generally, every p-norm on $\mathbb{R}^n$ is a monotone norm, see [Pli05, p.12], hence restricted to $\mathbb{R}_+^n$ they are monotone aggregation functions.

1.2.3. Matrix induced operators

In the preceding section we have considered general monotone operators on positive cones. Now we restrict our attention to a class of monotone operators that are induced by matrices with elements in $\mathcal{G}$.

An aggregation $\mu = (\mu_1, \ldots, \mu_n)^\top \in \mathrm{MAF}_n^n$, where $\mu_i \in \mathrm{MAF}_n$, $i = 1, \ldots, n$, is a map from $\mathbb{R}_+^{n \times n}$ to $\mathbb{R}_+^n$, defined by

$$\mu(A)_i = \mu_i(a_{i1}, \ldots, a_{in}), \quad \forall i = 1, \ldots, n,$$

for $A = (a_{ij})_{i,j=1}^n \in {\mathbb{R}_+}^{n \times n}$. A matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n \in \mathcal{G}^{n \times n}$ defines a map $\mathbb{R}_+^n \to {\mathbb{R}_+}^{n \times n}$ by

$$s \mapsto \Gamma(s) := (\gamma_{ij}(s_j))_{ij}$$

for $s \in \mathbb{R}_+^n$.

Let $\mu = (\mu_1, \ldots, \mu_n)^\top \in \mathrm{MAF}_n^n$ and $\Gamma = (\gamma_{ij})_{i,j=1}^n \in \mathcal{G}^{n \times n}$. Together these give rise to an operator $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by

$$\Gamma_\mu(s) := (\mu \circ \Gamma)(s) := \begin{pmatrix} \mu_1(\gamma_{11}(s_1), \ldots, \gamma_{1n}(s_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \ldots, \gamma_{nn}(s_n)) \end{pmatrix} \in \mathbb{R}_+^n$$

for $s \in \mathbb{R}_+^n$.

Lemma 1.2.3 *Let $\Gamma_\mu \in \mathcal{G}^{n \times n}$ and $\mu \in \mathrm{MAF}_n^n$. Then Γ_μ is monotone and satisfies $\Gamma_\mu(0) = 0$.*

Proof. We have $\Gamma_\mu(0) = 0 \in \mathbb{R}_+^{n\times n}$, since $\gamma_{ij}(0) = 0$ for any $i, j = 1, \dots, n$. So by (M1) $\Gamma_\mu(0) = 0$. Now let $u, v \in \mathbb{R}_+^n$ and consider

$$\begin{aligned}\Gamma_\mu(u+v) &= \begin{pmatrix} \mu_1(\gamma_{11}(u_1+v_1), \dots, \gamma_{1n}(u_n+v_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(u_1+v_1), \dots, \gamma_{nn}(u_n+v_n)) \end{pmatrix} \\ &\geq \begin{pmatrix} \mu_1(\gamma_{11}(u_1), \dots, \gamma_{1n}(u_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(u_1), \dots, \gamma_{nn}(u_n)) \end{pmatrix} = \Gamma_\mu(u),\end{aligned}$$

by monotonicity of γ_{ij} and μ_i, for $i, j = 1, \dots, n$. This proves monotonicity of Γ_μ. □

Important special cases of monotone aggregation functions μ are

1. $\mu_i(s) = \sum_{j=1}^n s_j$,
2. $\mu_i(s) = \max_{j=1}^n s_j$

for $s \in \mathbb{R}_+^n$ and $i = 1, \dots, n$. In the first case we write $\mu = \Sigma$ and in the second one $\mu = \oplus$. It will turn out that the case $\mu = \oplus$ has some very distinct properties and is in some sense simpler than the general case.

Let us also consider induced operators on subspaces spanned by a subset of the standard basis vectors: Let $I_1 \cup I_2 = \{1, \dots, n\}$ be such that $I_1 \cap I_2 = \emptyset$. Denote by $n_h = \#I_h$ the cardinality of I_h, $h = 1, 2$. An aggregation $\mu = (\mu_1, \dots, \mu_n)^\top \in \mathrm{MAF}_n^n$ induces maps from $\mathbb{R}_+^{n_h \times n_m}$ to $\mathbb{R}_+^{n_h}$, for $h, m = 1, 2$:

Denote $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$ and $A_{hm} = (a_{ij})_{i\in I_h, j\in I_m} \in \mathbb{R}_+^{n_h \times n_m}$, $h, m = 1, 2$. That is, up to renumbering of the standard basis vectors we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

For $h, m = 1, 2$ denote $\tilde{A}_{hm} = (\tilde{a}_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$,

$$\tilde{a}_{ij} = \begin{cases} a_{ij} & \text{if } i \in I_h \text{ and } j \in I_m, \\ 0 & \text{else,} \end{cases}$$

i.e., we have

$$\tilde{A}_{11} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{A}_{12} = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix},$$
$$\tilde{A}_{21} = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}, \qquad \tilde{A}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix},$$

and $A = \tilde{A}_{11} + \tilde{A}_{12} + \tilde{A}_{21} + \tilde{A}_{22}$. Then we define

$$\mu_{hm}(A_{hm}) := (\mu(\tilde{A}_{hm}))|_{I_h} := \begin{cases} (\mu(\tilde{A}_{hm}))_i & \text{if } i \in I_h, \\ 0 & \text{else,} \end{cases}$$

for $h, m = 1, 2$. The definition of μ_{hm} and subadditivity (M4) now yield

$$\mu(A) \leq \sum_{h,m=1}^{2} \mu_{hm}(A_{hm})$$

and in particular

$$(\mu(A))|_{I_h} \leq \mu_{h1}(A_{h1}) + \mu_{h2}(A_{h2}).$$

Example 1.2.4 *Let $A \in \mathbb{R}_+^{n\times n}$. Then the maps $t \mapsto a_{ij} \cdot t$ are of class $\mathcal{G}$. For $x \in \mathbb{R}_+^n$ we find $A_\Sigma(x) = Ax$, the familiar matrix-vector multiplication. A max-linear operator is $A_\oplus$ defined by $(A_\oplus(x))_i = \max_j a_{ij}x_j$.*

The last example shows, that two very well known kinds of operators belong to the class of operators

$$\{\Gamma_\mu : \Gamma_\mu \in \mathcal{G}^{n\times n}, \mu \in \mathrm{MAF}_n^n\},$$

namely monotone finite dimensional operators that are linear or max linear.

1.3. Graphs and projections

In the last subsection we associated monotone operators to $\mathcal{G}^{n\times n}$-matrices. As it is well known in matrix theory (see, e.g., [BP79]) there is a strong correspondence between directed graphs and nonnegative matrices in $\mathbb{R}^{n\times n}$. In this section we extend this correspondence to $\mathcal{G}^{n\times n}$-matrices.

Let I be an index set such that $\emptyset \neq I \subset \{1, \ldots, n\}$ and denote by $\#I$ the cardinality of I. Let $(e_k)_{k=1,\ldots,n}$ be the standard basis of $\mathbb{R}^n$. On $\mathbb{R}_+^n$ we denote by $\pi_I : \mathbb{R}_+^n \to \mathbb{R}_+^{\#I}$ the *projection* of the coordinates in $\mathbb{R}_+^n$ corresponding to the indices in I onto $\mathbb{R}^{\#I}$. The corresponding *anti-projection* is $\iota_I : \mathbb{R}_+^{\#I} \to \mathbb{R}_+^n$ mapping $x \mapsto (x_1 e_{i_1} + \ldots + x_{\#I} e_{i_{\#I}})$, where we write $I = \{i_1, \ldots, i_{\#I}\}$. For short we write π_i and ι_i instead of $\pi_{\{i\}}$ and $\iota_{\{i\}}$ for $i = 1, \ldots, n$. For any index set $\emptyset \neq I \subset \{1, \ldots, n\}$ and vector $x \in \mathbb{R}_+^n$ denote by $x|_I$ the vector in $\mathbb{R}_+^n$ with elements

$$(x|_I)_i = \begin{cases} x_i & \text{if } i \in I \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

A (finite) directed graph $G = \{V, E\}$ consists of a finite set of vertices V and a set of edges $E \subset V \times V$. In case of a graph with n vertices we may identify $V = \{1, \dots, n\}$. The adjacency matrix $A_G = (a_{ij})$ of this graph is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (j,i) \in E, \\ 0 & \text{else.} \end{cases}$$

Conversely, given an $n \times n$-matrix A, a graph $G(A) = \{V, E\}$ is defined by $V := \{1, \dots, n\}$ and $E = \{(j,i) \in V \times V : a_{ij} \neq 0\}$.

We say A_G is *irreducible*, if G is *strongly connected*, that is, for every pair of vertices (i,j) there exists a sequence of edges (a *path*), $(e_{i_0 i_1}, e_{i_1 i_2}, \dots, e_{i_{k-1} i_k})$, with $i_0 = i$, $i_k = j$ for some $k \geq 1$, and $e_{i_{l-1} i_l} \in E$ for $l = 1, \dots, k$, connecting vertex i to vertex j. Then k is the *length* of the path. Obviously A_G is irreducible if and only if its transpose $A_G^\top$ is. The adjacency matrix A_G is *reducible* if it is not irreducible. In that case A_G can be written as

$$P^\top \cdot \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \cdot P \tag{1.3}$$

where P is a permutation matrix, and A_{11} and A_{22} are square matrices. Speaking in terms of the graph G, the similarity transformation (i.e., multiplication of A_G with a permutation matrix) is a renumbering of the vertices.

The matrix A_G is *primitive*, if there exists a positive integer m such that $(A_G)^m$ has only positive entries. In terms of the graph G this means that there exists a number m, such that for any pair of vertices there exists a path of length m connecting these vertices.

Let $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a monotone operator. The operator T induces a graph $G = (V, E)$ on the set of vertices $V = \{1, \dots, n\}$ by the relation

$$(j,i) \in E \iff \pi_i \circ T \circ \iota_j \in \mathcal{K}_\infty.$$

The operator T will be called irreducible, reducible, or primitive, if A_G is any of these. See also the discussion in Section 1.6 for references to different definitions of graphs for monotone maps.

Let M be an $n \times n$ matrix, whose elements can be distinguished to be either positive or zero. For example, M could be a $\mathcal{G}^{n \times n}$-matrix. In this case positive entries correspond to strictly increasing functions, zero entries correspond to functions that are constantly zero, and negative entries do not occur. Define the adjacency matrix A of M by

$$a_{ij} = \begin{cases} 1 & \text{if } m_{ij} \text{ is positive,} \\ 0 & \text{else.} \end{cases} \tag{1.4}$$

The matrix M is called irreducible/ reducible/ primitive if and only if A is irreducible/ reducible/ primitive.

Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. The operator Γ_μ can be either irreducible or reducible. Note that this is in one-to-one correspondence with the same properties of the matrix Γ.

If Γ is reducible, then a permutation $P \in \{0,1\}^{n\times n}$, $P = (p_{ij})_{i,j=1}^n$, transforms it into a block upper triangular matrix $P\Gamma P^\top$, defined by $(P\Gamma P^\top)_{ij} = \sum_{k,l} p_{ki}p_{lj}\gamma_{kl}$. The blocks on the diagonal are either irreducible or zero 1×1-blocks. The transformation of Γ by a permutation matrix only rearranges the positions of the class $\mathcal{K}_\infty$-functions. The result is again a matrix in $\mathcal{G}^{n\times n}$. The product $P\Gamma P^\top = \tilde{\Gamma}$ is called *similarity transformation* of Γ.

In terms of operators a similarity transform is a change of coordinates, more particularly, a renumbering of the standard basis vectors. If we want to consider Γ_μ in the new coordinates, we also have to transform μ to $\tilde{\mu} := P\circ\mu\circ P^\top$. Here we identify the symbols P and $P^\top$ with their induced linear operators, i.e., $P^\top$ maps vectors $s \in \mathbb{R}_+^n$ to $P^\top s$. This yields an operator $\tilde{\Gamma}_{\tilde{\mu}}$, which is Γ_μ in the changed coordinates:

$$\tilde{\Gamma}_{\tilde{\mu}} = (P\circ\mu\circ P^\top)\circ(P\circ\Gamma\circ P^\top) = P\circ\Gamma_\mu\circ P^\top \tag{1.5}$$

Suppose Γ is irreducible, then again we have to distinguish two cases: Either Γ is primitive or it is not. In case it is not, the following result reveals some more detailed information.

Lemma 1.3.1 *Assume the matrix $\Gamma \in \mathcal{G}^{n\times n}$ is irreducible. Then there are two distinct cases:*

a) The matrix Γ is primitive and hence there is a non-negative integer k_0 such that Γ_μ^k satisfies $\pi_i\circ\Gamma_\mu^k\circ\iota_j \in \mathcal{K}_\infty$ for any $i,j\in\{1,\dots,n\}$ and $k \geq k_0$.

b) Using a permutation matrix P, the matrix Γ can be transformed to

$$P\Gamma P^\top = \begin{pmatrix} 0 & \tilde{\Gamma}_{12} & 0 & \dots & 0 \\ 0 & 0 & \tilde{\Gamma}_{23} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\Gamma}_{\nu-1,\nu} \\ \tilde{\Gamma}_{\nu 1} & 0 & 0 & \dots & 0 \end{pmatrix} =: \tilde{\Gamma}, \tag{1.6}$$

where the zero blocks on the diagonal are square. Denoting by A the adjacency matrix corresponding to the graph of $\tilde{\Gamma}$, we have that A^ν is of block diagonal form with square primitive blocks on the diagonal. The

number ν is termed index of imprimitivity *or* index of cyclicity*, which is the number of nonzero blocks in the above definition of* $\tilde{\Gamma}$*.*

Proof. Let A be the adjacency matrix corresponding to the graph associated with Γ_μ. This matrix is primitive if and only if Γ is primitive by the definition following (1.4). We now show that the $(i,j)^{\text{th}}$ entry of A^k is zero if and only if the $(i,j)^{\text{th}}$ entry of $A^{(k)}$, the adjacency matrix of the graph of $(\Gamma_\mu)^k$, is zero:

To this end note that $\pi_i \circ (\Gamma_\mu)^k \circ \iota_j \in \mathcal{K}_\infty$ if and only if there exist $i = l_1, \ldots, l_{k+1} = j \in \{1, \ldots, n\}$, such that $\pi_{l_m} \circ \Gamma_\mu \circ \iota_{l_{m+1}} \in \mathcal{K}_\infty$, for $m = 1, \ldots, k$. This follows easily by induction on k. Now $\pi_{l_m} \circ \Gamma_\mu \circ \iota_{l_{m+1}} \in \mathcal{K}_\infty$ if and only if $a_{l_m l_{m+1}} = 1$ and

$$(A^k)_{ij} = \sum_{\nu_1,\ldots,\nu_{k-1}=1}^{n} \underbrace{a_{i\nu_1} a_{\nu_1\nu_2} \cdots a_{\nu_{k-1}j}}_{\geq 0} \neq 0$$

if and only if there exists $i = l_1, \ldots, l_{k+1} = j \in \{1, \ldots, n\}$, such that $a_{l_m l_{m+1}} = 1$ for $m = 1, \ldots, k$. This proves the equivalence.

From these considerations it is clear, that it is sufficient to prove the lemma for the matrix A. For non-negative matrices this result is standard in the theory of non-negative matrices, see, e.g., [BP79, p.32, Theorem 2.2.20 and the sentence following its proof] or [LT85, p. 540, Theorem 15.4.2]. □

A result relating graph or, respectively, matrix structure to monotonicity is the next one.

Lemma 1.3.2 *Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. Then we have*

1. *Γ_μ is strictly increasing if and only if Γ has no zero rows; this is the case, e.g., for irreducible Γ;*

2. *Γ_μ is eventually strongly monotone if Γ is primitive.*

Proof. 1. Let $u, v \in \mathbb{R}_+^n$, $v \gg 0$. Then $u \ll u + v$ and if Γ has no zero rows

$$\Gamma_\mu(u+v) = \begin{pmatrix} \mu_1(\gamma_{11}(u_1+v_1), \ldots, \gamma_{1n}(u_n+v_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(u_1+v_1), \ldots, \gamma_{nn}(u_n+v_n)) \end{pmatrix}$$
$$\gg \begin{pmatrix} \mu_1(\gamma_{11}(u_1), \ldots, \gamma_{1n}(u_n)) \\ \vdots \\ \mu_n(\gamma_{n1}(u_1), \ldots, \gamma_{nn}(u_n)) \end{pmatrix} = \Gamma_\mu(u),$$

since for each row $i \in \{1, \ldots, n\}$ for at least one j the (i,j)th entry of Γ is of class $\mathcal{K}_\infty$, i.e., strictly increasing, so $\gamma_{ij}(u_j + v_j) > \gamma_{ij}(u_j)$. Now each μ_i is strictly monotone, hence $\mu_i(\ldots, \gamma_{ij}(u_j + v_j), \ldots) > \mu_i(\ldots, \gamma_{ij}(u_j), \ldots)$. Therefore Γ_μ is strictly increasing. If the ith row of Γ is a zero row, then there is no such (i,j)th entry of Γ of class $\mathcal{K}_\infty$. This proves the first part.

2. This is a direct consequence of Lemma 1.3.1a. □

1.4. Dynamical systems

According to [KH95, pp.1ff] the most general and somewhat vague notion of dynamical system includes the following ingredients:

- A "phase space" X, whose elements are points representing possible states of the system.
- "Time" set $\mathbb{T}$, which may be discrete or continuous. This is a directed set that either extends only into the future or also into the past. In the discrete-time case the time set is in a natural correspondence to the set of (nonnegative) integers. Similarly, in the continuous time case, time is represented by (nonnegative) real numbers.
- The time evolution law. In the most general case this is a law that determines the current state of the system depending on all its past states. Here we consider time evolution laws that determine the future by the actual state of the system. In other words the time evolution of a system depends only on the initial position of the system and the length of the evolution. Further we assume that the time evolution law does not change, i.e., is deterministic.

  We may assume that if the system is initially at a state $x \in X$, then after time t it is at a position only depending on t and x, but not on the moment in time when the system has been at x. This position can thus be denoted by $F(x,t)$. If we fix t, we obtain a transformation $\varphi^t : x \mapsto F(x,t)$. These transformations for different t are related to each other: The evolution for the state x for time $s+t$ can be accomplished by first applying φ^t to x and then φ^s to $\varphi^t(x)$. Thus we have $F(x, t+s) = F(\varphi^t(x), s)$, or equivalently, $\varphi^{t+s} = \varphi^t \circ \varphi^s$. This way dynamical systems define (semi) groups $\{F^n = (\varphi^1)^n | n \in \mathbb{Z}\}$ ($\mathbb{Z}$ replaced by $\mathbb{N}$) in the discrete time case and one-parameter (semi) groups $\{\varphi^t | t \in \mathbb{R}\}$ ($\mathbb{R}$ replaced by $\mathbb{R}_+$) in the continuous time case. This kind of systems is termed time invariant systems.

Note that it is also possible to have the time evolution law depend on time, which leads to time variant systems: If at time t_0 a system is in the state x_0, then after an evolution of length t it is at time $t_0 + t$ in a state that depends on x_0, t, and t_0. This state can be denoted by $G(x_0, t_0, t)$ and G satisfies for all x, t_0, t and s that $G(x_0, t_0, t + s) = G(G(x_0, t_0, t), t_0 + t, s)$.

Let X be a vector lattice with a positive cone X_+ with nonempty interior, see Section 1.2.1. Let $T : X_+ \to X_+$ be a monotone operator as introduced in Section 1.2.2. Consider the dynamical system

$$s_{k+1} = T(s_k), \quad k \geq 0. \tag{1.7}$$

The system (1.7) will be referred to as *(discrete time) monotone system*. The *orbit* of $x \in X_+$ is $O(x) := \{T^k(x)\}_{k\geq 0}$. As we have seen in Section 1.2.3 a special case of monotone systems are monotone systems with $T = \Gamma_\mu$ for some $\mu \in \mathrm{MAF}_n^n$ and $\Gamma \in \mathcal{G}^{n\times n}$.

Important examples of time variant continuous time systems are given by ordinary differential equations like

$$\dot{x} = f(t, x), \tag{1.8}$$

where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and locally Lipschitz continuous in x. This assumption locally guarantees existence and uniqueness of solutions to all initial conditions $x_0 \in \mathbb{R}^N$ and $t_0 \in \mathbb{R}$, see, e.g., [Aul97, Picard-Lindelöf Theorem, p. 68]. Solutions of (1.8) are denoted by $\varphi(x_0, t_0, \cdot)$ or $x(x_0, t_0, \cdot)$: If at time t_0 the system (1.8) started in x_0, then at time t it is in the state $\varphi(x_0, t_0, t)$. A trajectory is a function $t \mapsto \varphi(x_0, t_0, t)$.

The *omega limit set* of x is $\omega(x) := \bigcap_{k\geq 0} \overline{O(T^k(x))}$ in the case of discrete systems (1.7) and $\omega(x) := \bigcap_{t\geq t_0} \overline{\{\varphi(x, t_0, \tau) : \tau \geq t\}}$ for continuous time systems.

In the discrete time case, if $O(x)$ has compact closure and the operator T is continuous then $\omega(x)$ is nonempty, compact, invariant (that is $T(\omega(x)) = \omega(x)$), and invariantly connected, that is, $\omega(x)$ is not the union of two closed invariant sets, see [LaS76, Theorem 1.5.2, p.4].

If $T(x) = x$ or $f(t, x) = 0$ then x is an *equilibrium point*. The set of equilibrium points of (1.7) is denoted by E.

If $T^k(x) = x$ for some $k \geq 1$ then x is called *periodic* or *k-periodic*. The minimal such k is called the *period* of x (and $O(x)$).

A useful tool for monotone systems is the nonordering of periodic orbits theorem:

Theorem 1.4.1 ([HS05]) *A periodic orbit of a continuous monotone operator is unordered.*

Here unordered means that the elements of the periodic orbit are not comparable with respect to the partial order relation. Another fundamental result in the theory of monotone maps is the nonordering principle:

Theorem 1.4.2 ([HS05]) *Let $T : X_+ \to X_+$ be a continuous monotone operator and $x \in X_+$. Then*

1. *no points of $\omega(x)$ are related by $\ll$;*
2. *if $\omega(x)$ is a periodic orbit or T is SOP, no points of $\omega(x)$ are related by $<$.*

Both results have been proved in [HS05, Proposition 2.4 and Theorem 2.7].

As we will see later, the operators that we are concerned with in Chapter 2 will only have one equilibrium point: the origin. Also, they will not possess periodic orbits. So instead of periodic orbits we will be interested in stability properties of the the single equilibrium point.

An equilibrium point x is *stable* for system (1.7), if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for all y such that $\|x - y\| < \delta$,

$$\|x - T^k(y)\| < \varepsilon \text{ for } k \geq 0,$$

or, respectively, for system (1.8)

$$\|\varphi(y, t_0, t) - x\| < \varepsilon \text{ for } t \geq t_0.$$

A fixed point x is *(locally) attractive*, if there exists an $\varepsilon > 0$, such that for all y with $\|x - y\| < \varepsilon$,

$$\omega(y) = \{x\}.$$

The fixed point x is *globally attractive* if it is attractive for any $\varepsilon > 0$. An equilibrium point x is *globally asymptotically stable*, if x is globally attractive and stable.

Alternatively, stability concepts can be formulated using comparison functions, as, e.g., in [Hah67, p.7f], where a slightly different notion of class $\mathcal{K}$ and also the class $\mathcal{L}$ is used, which we did not introduce. Instead, we formulate the following Lemma:

Lemma 1.4.3 *An equilibrium point x is stable for system (1.7), if and only if there exists a $\mathcal{K}$ function ρ such that for all $\varepsilon > 0$ we have*

$$\|x - y\| < \rho(\varepsilon) \implies \|x - T^k(y)\| < \varepsilon \text{ for all } k \geq 0.$$

Proof. Clearly the existence of such a $\mathcal{K}$-function ρ implies stability of the equilibrium state x, as for any $\varepsilon > 0$, δ is directly given by $\rho(\varepsilon)$.

Conversely, assume that x is stable. Let $\alpha(\varepsilon)$ the 'maximal' possible δ from the stability definition, i.e., the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$, defined by

$$\alpha(\varepsilon) = \sup\{\delta > 0 : \|x - y\| < \delta \implies \|x - T^k(y)\| < \varepsilon \text{ for all } k \geq 0\}$$

for $\varepsilon > 0$ and $\alpha(0) = 0$. The function α is positive definite and hence it can be bounded from below by a class $\mathcal{K}$ function ρ with the desired properties. □

Similarly, global asymptotic stability of the origin for systems (1.7) and (1.8), respectively, is implied by the existence of a $\mathcal{KL}$ function β such that for all y and for all $k \geq 0$ we have

$$\|x - T^k(y)\| \leq \beta(\|x - y\|, k)$$

or, respectively, for all t_0, and $t \geq t_0$ we have

$$\|x - \varphi(y, t_0, t)\| < \beta(\|y - x\|, t - t_0).$$

Since $\beta(\cdot, 0)$ is of class $\mathcal{K}$, the stability part is at hand by the previous lemma. As for $t \to \infty$ the function $\beta(s, t)$ tends to zero for fixed s, attractivity of x is also obvious. The converse implication, however, is not true, since this particular formulation incorporates a uniformity of the attractivity, that is not guaranteed by global asymptotic stability in general. Nevertheless, it is indeed possible to formulate asymptotic stability in terms of comparison functions, see again [Hah67, p.8], but will not need that in the following.

1.5. Sets of decay and the "no joint increase" condition

Given a monotone operator $T : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}^n$ is endowed with the standard partial order induced by $\mathbb{R}^n_+$, we define the *sets of decay*

$$\Omega(T) = \{x \in \mathbb{R}^n_+ : T(x) \ll x\};$$
$$\Psi(T) = \{x \in \mathbb{R}^n_+ : T(x) \leq x\}.$$

For short we just write Ω or Ψ if the reference to T is clear from the context.

Note that in general $\Psi \neq \overline{\Omega}$, e.g., for the identity map. We always have $\Omega \subset \Psi$.

In this section and also later in Section 2.3 we have to be very specific about what we mean by unboundedness. We are going to need two kinds

of unboundedness, the second one being a stronger condition than the first one: We will call a set $A \in \mathbb{R}^n_+$ *radially unbounded* if for any $r > 0$ there exists an $x \in A \setminus B(0,r)$, where $B(0,r)$ denotes the open ball in $\mathbb{R}^n_+$ of radius r centered at the origin. This definition agrees with what is usually called 'unbounded'. We will call A *jointly unbounded* if for every $x \in \mathbb{R}^n_+$ there exists an $y \in A$ such that $y \geq x$.

Example 1.5.1 *For $T = \frac{1}{2} id$ on $\mathbb{R}^n_+$ we have $\Psi = \mathbb{R}^n_+$ and $\Omega = \operatorname{int} \mathbb{R}^n_+$. Both sets are radially and jointly unbounded.*

A geometrically interesting but rather obvious result is the next one. It states that the *no joint increase condition*,

$$\forall s \neq 0 : T(s) \not\geq s,$$

holds if and only if the map T always decays in at least one component (except for the origin). More rigorously, for $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ define the sets

$$\Omega_i := \Omega_i(T) := \{s \in \mathbb{R}^n_+ : s_i > \pi_i \circ T(s)\}.$$

So by definition we have $\Omega = \bigcap_{i=1}^n \Omega_i$. Now we relate the topological properties of the sets Ω_i, $i = 1, \ldots, n$, to the algebraic properties of T.

Lemma 1.5.2 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Then $T(x) \not\geq x$ for all $x \neq 0$ if and only if*

$$\left(\bigcup_{i=1}^n \Omega_i\right) \cup \{0\} = \mathbb{R}^n_+.$$

A more involved result concerns the intersection of the sets Ω_i. Define

$$S_r := \{s \in \mathbb{R}^n_+ : \sum_i s_i = r\}$$

to be the $n-1$-simplex in $\mathbb{R}^n_+$ of distance r to the origin. This simplex will play an important role in the sequel.

Proposition 1.5.3 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous. Let $r > 0$ and assume that $T(x) \not\geq x$ for all $x \in S_r$. Then*

$$\left(\bigcap_{i=1}^n \Omega_i\right) \cap S_r \neq \emptyset,$$

and $x \in \bigcap_{i=1}^n \Omega_i \cap S_r$ implies $x \gg 0$. In particular, if $T(x) \not\geq x$ for all $x \neq 0$ then the set Ω is radially unbounded.

A proof will be given later in this section.

Example 1.5.4 *Let us consider a numerical example, to gain some intuition. Let*

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{pmatrix} \in \mathcal{G}^{3\times 3}$$

and let $\mu = \oplus$, *with*

$$\gamma_{12}(t) = t^2/m^2 \qquad \gamma_{13}(t) = t^2/m^2$$
$$\gamma_{21}(t) = \sqrt{t}/p_2 \qquad \gamma_{23}(t) = t/q_2$$
$$\gamma_{31}(t) = \sqrt{t}/p_3 \qquad \gamma_{32}(t) = t/q_3$$

for

$$m = 9/15$$
$$p_2 = 5 \qquad p_3 = 4$$
$$q_2 = 8/3 \qquad q_3 = 11/10.$$

For $r = 0.06$ *we indicate the decay sets* $\Omega_i \cap S_r$ *for* $i = 1, 2, 3$ *in Figure 1.2. In fact, it is possible to check numerically and also analytically that* $\Gamma_\oplus(s) \not\geq s$ *for all* $s \in \mathbb{R}^n_+$, $s \neq 0$, *see Lemma 2.3.14 for an analytical approach and Section 4.1.1 for the numerical approach.*

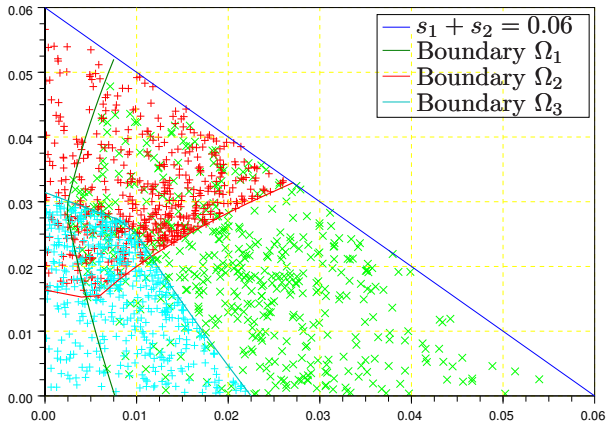


Figure 1.2.: The intersection of the decay sets Ω_i, $i = 1, 2, 3$, on S_r for $r = 0.06$ in Example 1.5.4. Here the boundaries of the sets Ω_i can be calculated explicitly and are indicated in the figure. The little crosses are some random points colored according to the boundary of the set they are contained in.

An almost immediate consequence is that the assertion of Proposition 1.5.3 is also true for powers of T. To prove this, we state the following lemma:

Lemma 1.5.5 *Let* $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *be a monotone continuous operator satisfying* $T(x) \not\geq x$ *for all* $x \neq 0$. *Then* $T^k(x) \not\geq x$ *for all* $x \neq 0$ *and* $k \geq 1$.

Proof. Assume the contrary: Suppose there exist an $x \neq 0$ and a $k > 1$, such that $T^k(x) \geq x$. Let $z = \sup\{x, T(x), \ldots, T^{k-1}(x)\}$. Clearly $z \neq 0$. Hence

$$T(z) \geq \sup\{T(x), \ldots, T^k(x)\}$$

by monotonicity and

$$= \sup\{x, T(x), \ldots, T^k(x)\}$$

since $T^k(x) \geq x$, which also implies

$$\geq \sup\{x, T(x), \ldots, T^{k-1}(x)\} = z$$

contradicting $T(z) \ngeq z$ for all $z \neq 0$. Hence no such x and k exist, i.e., for all $k \geq 1$ and $x \neq 0$ we have $T^k(x) \ngeq x$. □

Corollary 1.5.6 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be continuous and monotone. Then $T(x) \ngeq x$ for all $x \neq 0$ implies that for any $r > 0$, $k \geq 1$,*

$$\Omega(T^k) \cap S_r \neq \emptyset.$$

Proof of Corollary 1.5.6. By Lemma 1.5.5 we have $T^k(x) \ngeq x$ for all $x \neq 0$ and $k \geq 1$. So we just apply Proposition 1.5.3 to T^k. □

For the proof of Proposition 1.5.3 we need the famous Knaster-Kuratowski-Mazurkiewicz (KKM) theorem. To state the theorem we need to introduce some notation. In $\mathbb{R}^{n+1}_+$ let $\{e_i\}_{i=1}^{n+1}$ be the standard basis of $\mathbb{R}^{n+1}_+$. Then by $S_1 = \text{co}\{e_i\}_{i=1}^{n+1}$, the convex hull of the unit vectors, we denote the *unit n-simplex*. This simplex has $n+1$ *vertices*, namely e_i, $i = 1, \ldots, n+1$. A *face* is a simplex spanned by a subset of the vertices of S_1, i.e., for $I \subset \{1, \ldots, n+1\}$ we call $\sigma = \text{co}\,(\{e_i\}_{i \in I})$ a face of S_1. By I_σ we denote the indices of the vertices spanning σ, in this case this is the set I.

Theorem 1.5.7 (Knaster-Kuratowski-Mazurkiewicz, 1929) *Let S_1 denote the unit n-simplex, and for a face σ of S_1 let I_σ denote the indices of the vertices of σ.*

If a family $\{A_i\}_{i=1,\ldots,n+1}$ of subsets of S_1 is such that all the sets A_i are closed or all are open, and each face σ of S_1 is contained in the corresponding union $\bigcup_{i \in I_\sigma} A_i$, then

$$\bigcap_{i=1}^{n+1} A_i \neq \emptyset.$$

The original proof for closed sets has been given in [KKM29], while the formulation above is taken from [HL97] and has been proved in [Las90].

Proof of Proposition 1.5.3. Note that S_r for $r > 0$ is a $(n-1)$-simplex with vertices $r \cdot e_i$, $i = 1, \ldots, n$. We will show that each (nonempty) face σ spanned by $r \cdot e_i$, $i \in I_\sigma \subset \{1, \ldots, n\}$, satisfies the assumptions of the Knaster-Kuratowski-Mazurkiewicz theorem with respect to the sets $A_i = \Omega_i$, $i = 1, \ldots, n$, i.e.,

$$\sigma = \mathrm{co}\{r \cdot e_i : i \in I_\sigma\} \subset \bigcup_{i \in I_\sigma} (\Omega_i \cap S_r)\,. \tag{1.9}$$

To this end let $x \in \sigma$. Then $x = x|_{I_\sigma}$, i.e., all components x_i, $i \notin I_\sigma$ are zero. Now $T(x) \not\geqq x$ implies $(T(x))_i < x_i$ for some $i \in \{1, \ldots, n\}$. Suppose $i \notin I_\sigma$, then $(T(x))_i < x_i = 0$, contradicting that the image of T lies in $\mathbb{R}^n_+$. Hence $i \in I_\sigma$ and so $x \in \Omega_i \cap S_r \subset \bigcup_{I_\sigma}(\Omega_i \cap S_r)$, proving (1.9). The sets

$$\Omega_i = \{s \in \mathbb{R}^n_+ : \pi_i \circ T(s) < s_i\},$$

$i = 1, \ldots, n$ are all open by continuity of T. This is also true with respect to the relative topology in S_r. Now the KKM-theorem implies that $\bigcap_{i=1}^n (\Omega_i \cap S_r) \neq \emptyset$. Clearly this makes Ω radially unbounded if $T(x) \not\geqq x$ for all $x \neq 0$.

Finally, pick $x \in \bigcap_{i=1}^n (\Omega_i \cap S_r)$. In particular $x \in \Omega_i$ for all $i = 1, \ldots, n$, so that $0 \leq \pi_i \circ T(x) < x_i$. This implies $x \gg 0$. □

Now we give two catalogs of properties of the sets Ψ and Ω. These catalogers will help to better understand geometric properties of Ψ and Ω, especially when T has more "structure" than just being monotone.

Lemma 1.5.8 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone. Then we have*

1. *$T^{k+1}(\Psi) \subset T^k(\Psi) \subset \Psi$ for $k \geq 0$;*
2. *$T^k(\Omega) \subset \Omega$ for $k \geq 0$ if T is strictly increasing;*
3. *$T^k(\Psi \setminus \{0\}) \subset \Omega$ for $k \geq 1$ if T is strongly monotone and $T(x) \neq x$ for all $x \neq 0$;*
4. *there exists a $k_0 \geq 1$ such that $T^k(\Psi \setminus \{0\}) \subset \Omega$ for $k \geq k_0$ if T is eventually strongly monotone.*

Proof. 1. Let $x \in \Psi$, i.e., $T(x) \leq x$. By monotonicity of T we find $T(T(x)) \leq T(x) \leq x$ and inductively $T^{k+2}(x) \leq T^{k+1}(x) \leq T^k(x) \leq x$ for all $k \geq 0$, i.e. $T^{k+1}(x) \in T^k(\Psi)$ and $T^k(x) \in \Psi$.

2. Let $x \in \Omega$, then $T(x) \ll x$. Applying the strictly increasing operator T gives $T^2(x) \ll T(x) \ll x$ and inductively $T(T^k(x)) \ll T^k(x)$, i.e., $T^k(x) \in \Omega$.

3. Let $x \in \Psi$, $x \neq 0$. Then $0 < x$ and $0 \leq T(0) \ll T(x) < x$ as $T(x) \neq x$. For $k \geq 1$ an application of the strongly monotone operator T^k to $0 \ll T(x) < x$ gives $T(T^k(x)) \ll T^k(x)$, that is, we have $T^k(x) \in \Omega$.

4. Since T is eventually strongly monotone, there exists a $k_0 \geq 1$ such that $x < y$ implies $T^k(x) \ll T^k(y)$. The rest of this proof goes along the lines of part 3 for $k \geq k_0$.

□

The second catalog involves a diagonal operator D. Such diagonal operators will play a crucial role in Section 2.3.

Lemma 1.5.9 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone. Let $\rho \in \mathcal{K}_\infty$, $\rho > id$, and $D = \operatorname{diag}(\rho)$. Then we have*

1. *$\Psi(D \circ T) \setminus \{0\} \subset \Omega(T)$, if T is strongly monotone;*

2. *$\Psi(D \circ T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\} \subset \Omega(T)$, if T is strictly increasing;*

3. *$T\left(\Psi(D \circ T) \setminus \{0\}\right) \subset \Omega(T)$, provided that T is strongly monotone.*

All results are also true with $D \circ T$ replaced by $T \circ D$.

Proof. 1. Let $x \in \Psi(D \circ T) \setminus \{0\}$. Then $0 < x$ implies $0 \ll T(x) \ll D(T(x)) \leq x$ and hence $x \in \Omega(T)$. Now let $x \in \Psi(T \circ D) \setminus \{0\}$. Since $0 < x < D(x)$ we have $0 \ll T(x) \ll T(D(x)) \leq x$, so $x \in \Omega(T)$.

2. Let $x \in \Psi(T \circ D) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\}$. Again $0 \ll x \ll D(x)$ implies $0 \ll T(x) \ll T(D(x)) \leq x$, hence $x \in \Omega(T)$. Let $x \in \Psi(D \circ T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\}$. Then $0 \ll x$ gives $0 \ll T(x) \ll D(T(x)) \leq x$, hence $x \in \Omega(T)$.

3. This result is implied by 1. and Lemma 1.5.8.2. □

In general the preimage $T^{-1}[\Psi]$ is not a subset of Ψ:

Example 1.5.10 *Let $A = \begin{pmatrix} 0.2 & 0 \\ 0.8 & 0.3 \end{pmatrix}$, so we have $\rho(A) = 0.3$. For $x = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ we find $Ax = \begin{pmatrix} 0.1 \\ 0.55 \end{pmatrix} \not\leq x$, but $A^2x = \begin{pmatrix} 0.02 \\ 0.245 \end{pmatrix} \ll Ax$. Hence $Ax \in \Psi$ but $x \notin \Psi$.*

1.6. Notes and references

The notation of partial order relations has been taken from the book by Smith [Smi95]. The different notions of monotonicity come from [HS05], with the exception of strictly increasing. The property that we call strictly increasing does not seem to play a role in the cited literature. Nevertheless we find this property interesting, as it is natural for operators Γ_μ with $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$, such that Γ has no zero rows. Also it is a weaker notion than strongly monotone. In [XO06] and [Ama72] strictly increasing is the same as our strictly monotone. The notions concerning lattices have been taken from [SW99].

The definition of classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ are standard in control literature, see, e.g., the works by Sontag; Jiang, Teel, and Praly; Jiang, Mareels, and Wang. According to [Grü02a, p.201] and [Grü04], this naming for the sets $\mathcal{K}$ and $\mathcal{KL}$ has been first introduced in Hahn [Hah59, Hah67]. Apparently Hahn gives no reasons for the nomenclature, but according to rumors $\mathcal{K}$ might derive from Kamke, see [Grü02a, p.201]. Class $\mathcal{G}$ derives its name from *gain*, which will seem natural in the chapter on input-to-state stability. For a good starting point on spectral properties and stability of max-linear operators see [Lur05].

The definition of a graph of a monotone operator is somewhat arbitrary. We chose the current approach because of its nice interactions with operators Γ_μ with $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$, that are of special interest in the next chapter.

Note that if the graph of a monotone operator T lacks the edge (i, j), then this does not mean that the jth argument does not influence the ith component of T. There do exist other notions of graphs of monotone operators, see for instance [AS04a, AS04b, GG04]. Note that the graph theory we used here can be found in the literature on (nonnegative) matrix theory, see for instance [LT85, BP79].

The definition of dynamical systems is taken from [KH95]; the stability concepts as introduced in Section 1.4 are based on [HP05].

The sets of decay Ω and Ψ and the no joint increase condition

$$T(x) \ngeq x, \quad \text{for all } x \neq 0$$

first appeared in Dashkovskiy et al. [DRW05, DRW07] in the context of large scale interconnections of input-to-state stable control systems. The relation between the no joint increase condition and the decay sets has already been discovered therein.

2. Monotone dynamics

In this chapter we investigate dynamical systems induced by a monotone operator T on the cone $\mathbb{R}^n_+$. For certain such operators it turns out that the stability properties of the origin with respect to the monotone dynamical system

$$s_{k+1} = T(s_k), \quad k \geq 0,$$

are in strong correspondence to the order condition

$$\forall s \neq 0 : T(s) \not\geq s.$$

In Section 2.1 we consider monotone operators without a special structure, whereas in Section 2.2 we derive special results for those monotone operators, that are given by matrices of comparison functions and monotone aggregation functions. In Section 2.3 these operators turn out to give the most fruitful theory, especially if the monotone aggregation function is $\oplus$.

Probably the best motivation for this chapter is the linear case: Let ρ denote the spectral radius of $A \in \mathbb{R}^{n\times n}$, i.e.,

$$\rho(A) = \max\{|\lambda| : \ \lambda \text{ is an eigenvalue of } A\}.$$

Lemma 2.0.1 *Let $A \in \mathbb{R}^{n\times n}_+$. Then the following are equivalent:*

(1) $Ax \not\geq x$ for all $x \in \mathbb{R}^n_+, x \neq 0$;

(2) $\rho(A) < 1$;

(3) there exists a vector $s \gg 0$ such that $As \ll s$;

(4) the origin is globally asymptotically stable for the dynamical system defined by

$$s_{k+1} = As_k, \quad k \geq 0.$$

If any of the above conditions is satisfied, then an important consequence is that $(id - A)$ is invertible; hence $x = Ax + b$ is uniquely solvable for any $b \in \mathbb{R}^n_+$; furthermore all solutions x of $(id-A)x = b$ for $b \geq 0$ are nonnegative.

Proof. Most of this is standard, we state a few references for the sake of completeness. It is well known that $A^k \to 0$ as $k \to \infty$ if and only if $\rho(A) < 1$. This gives the equivalence of (4) and (2). The theory for the equivalences (1) $\iff$ (2) $\iff$ (3) can be found in [BP79] as follows:

(1) $\iff$ (2) and (3) $\implies$ (1) is [BP79, Theorem 2.1.11, Corollary 2.1.12, p. 28]. To this end note that the implication

$$Ax \geq \alpha x \text{ for some } x > 0 \implies \rho(A) \geq \alpha$$

yields by the law of contraposition

$$\rho(A) < \alpha \implies \text{there exists no } x > 0, \text{ so that } Ax \geq \alpha x$$

where the latter can be stated equivalently as

$$Ax \ngeq \alpha x \text{ for all } x > 0.$$

And this is just (1).

As we discuss in in Section 2.3, there exists a vector $s \gg 0$ such that $As \ll s$. Alternatively, the implication (1) $\implies$ (3) follows easily from Proposition 1.5.3.

The Theorem by Neumann (see, e.g., [RY00, Theorem 4.40, p. 113]) states that the inverse of $(\mathrm{id} - A)$ is given by

$$(\mathrm{id} - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

if $\rho(A) < 1$. From the right hand side it is clear that solutions x to $(\mathrm{id}-A)x = b$ are nonnegative if $b \geq 0$. □

2.1. From attractivity to order

Throughout this section we fix the following setting: Let X be a vector lattice with closed partial order $\leq$ induced by a positive cone X_+ with nonempty interior.

For a monotone operator $T : X_+ \to X_+$ we investigate the relations between

1. the *no joint increase condition*

$$\forall s \neq 0 : T(x) \ngeq s, \tag{2.1}$$

and

2. global asymptotic stability of the origin for the dynamical system

$$s_{k+1} = T(s_k), \quad k \geq 0. \tag{2.2}$$

It turns out that attractivity of the origin for (2.2) implies condition (2.1), see Corollary 2.1.2. But first we only assume that the origin is locally attractive.

Proposition 2.1.1 *Let $T : X_+ \to X_+$ be a monotone operator. Assume that 0 is locally attractive with respect to system (2.2). Then the region of attraction of the origin is contained in the set $\{x \in \mathbb{R}^n_+ : T(x) \not\geq x\}$.*

In particular there exists a neighborhood U of the origin, so that $T(x) \not\geq x$ for all $x \in U$, $x \neq 0$.

Proof. The region of attraction of the origin is the set V of points $x \in X_+$ such that $T^k(x) \to 0$ as $k \to \infty$.

Now we argue indirectly. Let $x \in V$, $x > 0$, and assume $T(x) \geq x$. Then by monotonicity $T^{k+1}(x) \geq T^k(x) \geq x$ for all $k \geq 0$. Hence the orbit $O(x)$ is not attracted to 0, contradicting our assumption that $x \in V$. Therefore we must have $T(x) \not\geq x$ for all $x \in V$, $x \neq 0$.

In particular, since the origin is locally attractive, V contains a neighborhood U of the origin. □

More interestingly for us now is the global version of this result, when the origin is globally attractive.

Corollary 2.1.2 *Let $T : X_+ \to X_+$ be a monotone operator. Assume that 0 is globally attractive with respect to system (2.2). Then $T(x) \not\geq x$ for all $x \neq 0$.*

Another interesting consequence is the next result, as it states that for the class of monotone maps under consideration, the origin is stable if it is attractive. For general dynamical systems this is known not to be the case, see [Hah67, pp. 191–194].

Lemma 2.1.3 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous. Suppose that the origin is locally attractive with respect to system (2.2). Then the origin is also stable.*

Proof. By Proposition 2.1.1 there exists a neighborhood U of the origin, so that $T(x) \not\geq x$ for all $x \in U$, $x \neq 0$. Hence for a given $\varepsilon > 0$ we may pick an $r \in (0, \varepsilon]$ small enough such that $S_r \subset U$ and by Proposition 1.5.3 choose a point $y \in \Omega(T) \cap S_r$. This point satisfies $y \gg 0$. Let

$$\delta := \sup\{d \in \mathbb{R}_+ : x \ll y \ \forall x \in B(0, d)\}.$$

Note that $\delta > 0$ and that $\|x\| < \delta$ implies $x \ll y$. By monotonicity this implies $T^k(x) \leq T^k(y) \ll y$ for $k \geq 1$, hence $\|T^k(x)\|_1 < r \leq \varepsilon$, proving stability of the origin. □

A map satisfying condition (2.1) can have only 0 as a fixed point. The next result states that indeed 0 must be a fixed point.

Lemma 2.1.4 *Let $T : X_+ \to X_+$ be a monotone operator satisfying $T(x) \not\geq x$ for all $x \neq 0$. Then $T(0) = 0$.*

Proof. Suppose $T(0) = x > 0$. Then by monotonicity $T(x) \geq T(0) = x > 0$ contradicting $T(x) \not\geq x$ for $x \neq 0$. Hence we must have $T(0) = 0$. □

We are now able to establish the following qualitative equivalence.

Theorem 2.1.5 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be continuous and monotone. Then the origin is locally asymptotically stable for the discrete time system (2.2) if and only if T satisfies $T(x) \not\geq x$ for all $x \neq 0$ in a neighborhood of the origin.*

Proof. If the origin is locally asymptotically stable, then the operator T satisfies $T(x) \not\geq x$ for all $x \neq 0$ in a neighborhood of the origin by Proposition 2.1.1.

Conversely, let us suppose that there exists a neighborhood U of the origin where T satisfies $T(x) \not\geq x$ for all $x \in U$, $x \neq 0$. Next we show that the origin is attractive for system (2.2), so that local asymptotic stability follows by application of Lemma 2.1.3. To this end, for $\varepsilon > 0$ chosen sufficiently small, we have $S_\varepsilon = \{x \in \mathbb{R}^n_+ : \|x\|_1 = \varepsilon\} \subset U$. For example, ε can be chosen such that the closed ε-ball with respect to the 1-norm is contained in U. By Proposition 1.5.3 we find a point $y \gg 0$, $y \in \Omega \cap S_\varepsilon$.

By definition $T(y) \ll y$ and by monotonicity and repeated application of T we find that $T^{k+1}(y) \leq T^k(y) \ll y$ is a bounded and decreasing sequence, hence convergent. The limit point y^* is a fixed point of T, i.e., satisfies $T(y^*) = y^* \ll y$, i.e., we have $y^* \in U$. Suppose $y^* \neq 0$. Since $T(x) \not\geq x$ for all $x \in U$, $x \neq 0$, the point y^* cannot be a fixed point, a contradiction. Hence we must have $y^* = 0$.

The same argument works for every trajectory starting in the order interval $V := [0, y[$, as this trajectory, too, is bounded above by y due to monotonicity of T.

The order interval V is a neighborhood of the origin, so we proved that the origin is locally attractive with a basin of attraction that at least contains the set V. □

The question we considered initially regards the corresponding global version of this theorem. By reconsidering Lemma 2.0.1 we see that the converse

of Corollary 2.1.2 holds for linear monotone (finite-dimensional) maps. The question imposing itself now is, if the converse of Corollary 2.1.2 holds in general, i.e., on a global scale. It does not, as the next example shows.

Example 2.1.6 *Fix the real constants $\lambda \in]0,1[$ and $\mu \geq 0$. Let $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be given by*

$$T\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \lambda s_1 + s_1^2 s_2 + \mu s_2 \\ \lambda s_2 \end{pmatrix}$$

for all $s = (s_1, s_2)^\top \in \mathbb{R}^2_+$.

The map T satisfies $T(0) = 0$ and is continuous and monotone. It satisfies $T(s) \not\geq s$, for all $s \neq 0$. To see this, consider the cases $(s_1 \geq 0, s_2 > 0)$ and $s_2 = 0$ separately. If $s^0 = (s_1^0, s_2^0)^\top > 0$ is such that $s_1^0 > 1/(\lambda s_2^0)$, then the trajectory of (2.2) starting in s^0 is unbounded in the first component: The condition $s_1(k) > 1/(\lambda s_2(k))$ with $s(k) > 0$ implies $s_1(k+1) = s_1(k)(\lambda + s_1(k)s_2(k) + \mu s_2(k)) > s_1(k)/\lambda > 1/(\lambda^2 s_2(k)) = 1/(\lambda s_2(k+1))$ and clearly $s_2(k+1) > 0$. By induction we obtain a trajectory that converges to 0 in the second component and diverges in the first one. Hence for the monotone system induced by T the origin is not globally asymptotically stable. The picture in Figure 2.1 is drawn for $\lambda = 1/2$ and $\mu = 1/16$.

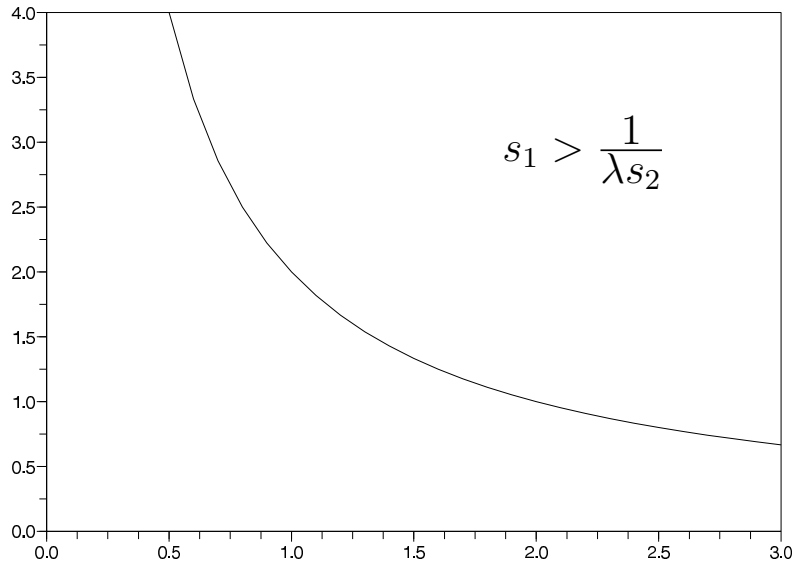


Figure 2.1.: The repelling set in Example 2.1.6

Note that in Example 2.1.6 the map T cannot be written in the form Γ_μ for some $\mu \in \mathrm{MAF}_n^n$ and $\Gamma \in \mathcal{G}^{n\times n}$, as the term $s_1^2 s_2$ is not compatible with this condition. This is essentially because a function of the form $(a, b) \mapsto a + a^2 b + b$ is not subadditive on $\mathbb{R}^2_+$. We may conjecture now, that the matrix structure

of T plays a crucial role for the attractivity of the origin. We will investigate the consequences of T having a matrix structure in the next section.

2.2. From order to attractivity

In this section we consider monotone operators that are induced by $\mathcal{G}^{n\times n}$-matrices. Let $\mu \in \mathrm{MAF}_n^n$ and $\Gamma \in \mathcal{G}^{n\times n}$. Consider $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ and suppose $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$. Is the origin globally attractive or even globally asymptotically stable for the system

$$s_{k+1} = \Gamma_\mu(s_k), \quad k \geq 0, \tag{2.3}$$

for such kind of operators? Or do we have to impose even more restrictive assumptions on Γ_μ to yield attractivity of the origin?

A first result towards the answer is the next result.

Theorem 2.2.1 *Let $\mu \in \mathrm{MAF}_n^n$ and $\Gamma \in \mathcal{G}^{n\times n}$. Assume $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ satisfies $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$ and assume further that Γ is irreducible. Then the origin is globally asymptotically stable for system (2.3).*

In order to prove this theorem we need two auxiliary lemmas.

Lemma 2.2.2 *Let $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a continuous and monotone operator satisfying $T(x) \not\geq x$ for all $x \neq 0$. Then for any $x \in \mathbb{R}_+^n$ such that $O(x)$ has compact closure, $\omega(x) = \{0\}$.*

Proof. Clearly $\omega(x)$ is nonempty and it is compact, see [LaS76, Theorem 1.5.2, p.4]. Hence $z := \sup \omega(x)$ is well defined. By invariance we have $T(z) \geq T(\omega(x)) = \omega(x)$ and hence $T(z) \geq \sup \omega(x) = z$. This implies $z = 0$, hence $\omega(x) = \{0\}$. □

Another ingredient is the following easy to believe fact, that also holds in a more general setting.

Lemma 2.2.3 *Let X be a finite dimensional Banach space. Let $T : X \to X$ be a continuous map and let $k_0 \geq 1$. Then for a given $x \in X_+$ the orbit $O(x) = \{T^k(x)\}_{k\geq 0}$ is bounded if and only if the orbit $O_{k_0}(x) := \{T^{j\cdot k_0}(x)\}_{j\geq 0}$ is bounded.*

Proof. First assume that $O(x)$ is bounded. Clearly any sequence in $O(x)$ must be bounded, too, proving that $O_{k_0}(x)$ must be bounded.

Assume that $O_{k_0}(x)$ is bounded. Then $\overline{O_{k_0}(x)}$ is compact. The maps $T, T^2, \ldots, T^{k_0-1}$ are all continuous. The continuous image of a compact set is compact, see [Arm83, Theorem 3.4, p.47], hence bounded, since X is

finite dimensional, and the finite union of bounded sets is bounded. Hence $O(x) \subset \bigcup_{j=0}^{k_0-1} T^j(\overline{O_{k_0}(x)})$ must be bounded. □

Proof of Theorem 2.2.1. By Lemma 1.3.1 there are two main cases: Either Γ is primitive or it is not.

Let us first assume that Γ is primitive. We argue by contradiction. Suppose there exists an $x \in \mathbb{R}^n_+$ such that $\limsup_{k\to\infty} \|\Gamma^k_\mu(x)\| = \infty$ and $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$.

By Lemma 1.3.1 there exists a $k_0 \geq 1$, such that Γ^k_μ satisfies $\pi_i \circ \Gamma^k_\mu \circ \iota_j \in \mathcal{K}_\infty$ for any $i, j \in \{1, \dots, n\}$ and $k \geq k_0$. Hence for every $i = 1, \dots, n$, there exists an $M_i \in \mathbb{R}$, $M_i > 0$, such that for all $t \in \mathbb{R}$, $t > M_i$, we have $\Gamma^{k_0}_\mu(t \cdot e_i) \gg x$. Since $\limsup_{k\to\infty} \|\Gamma^k_\mu(x)\| = \infty$ there exists a $k_1 \in \mathbb{N}$ and index i_0 such that

$$\pi_{i_0}(\Gamma^{k_1}_\mu(x)) > M_{i_0}.$$

By monotonicity of Γ_μ we have

$$\begin{aligned}\Gamma^{k_0}_\mu \circ \Gamma^{k_1}_\mu(x) &\geq \max_{i\in\mathbb{N}} \Gamma^{k_0}_\mu(\pi_i(\Gamma^{k_1}_\mu(x)) \cdot e_i) \\ &\geq \Gamma^{k_0}_\mu(\pi_{i_0}(\Gamma^{k_1}_\mu(x)) \cdot e_{i_0}) \geq \Gamma^{k_0}_\mu(M_{i_0} \cdot e_{i_0}) \geq x.\end{aligned}$$

This contradicts $\Gamma^k_\mu(s) \not\geq s$ for all $k \geq 1$ and $s \in \mathbb{R}^n_+$, which is asserted by Lemma 1.5.5.

Now the orbit $O(x)$ is bounded, by Lemma 2.2.2 we find that the origin is globally attractive.

This proves attractivity of the origin for primitive Γ.

Now assume that the index of imprimitivity ν, given by Lemma 1.3.1, is greater than 1. There exists a matrix $P \in \{0,1\}^{n\times n}$ such that

$$P\Gamma P^\top = \begin{pmatrix} 0 & \tilde{\Gamma}_{12} & 0 & \dots & 0 \\ 0 & 0 & \tilde{\Gamma}_{23} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\Gamma}_{\nu-1,\nu} \\ \tilde{\Gamma}_{\nu 1} & 0 & 0 & \dots & 0 \end{pmatrix} =: \tilde{\Gamma}. \tag{2.4}$$

The similarity transformation $\tilde{\Gamma}$ of Γ corresponds to a renumbering of the standard basis vectors (or to a renumbering of the vertices in the graph G_Γ). Therefore for the analysis of attractivity of the origin or boundedness of orbits we may as well consider $\tilde{\Gamma}$ instead of Γ. Of course we also have to redefine

μ to $\tilde{\mu} = P \circ \mu \circ P^\top$, as in Section 1.3, just before Lemma 1.3.1. So without loss of generality we assume that Γ is already of the form depicted in (2.4).

Hence there exist ν disjoint index sets $I_1, \dots, I_\nu$, with $\bigcup_i I_i = \{1, \dots, n\}$, $n_i := \#I_i \geq 1$, such that $\pi_{I_i} \circ \Gamma_\mu^\nu \circ \iota_{I_i} : \mathbb{R}_+^{n_i} \to \mathbb{R}_+^{n_i}$ are primitive, $i = 1, \dots, n$. From here we see, that the remaining argument reduces to a ν-times simultaneous application of the first part of the proof to the primitive blocks $\pi_{I_i} \circ \Gamma_\mu^\nu \circ \iota_{I_i}$. This at first only proves that orbits for the map Γ_μ^ν are bounded. By Lemma 2.2.3 also the orbits of the map Γ_μ must be bounded. Attractivity of the origin now follows as in the primitive case. Stability then is a consequence of Lemma 2.1.3. This completes the proof of Theorem 2.2.1. □

If we know that the sets of decay are jointly unbounded, then we immediately obtain a result on the attractivity of the origin:

Lemma 2.2.4 *Let $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be continuous and monotone, such that $T(x) \not\geq x$ for all $x \neq 0$ and assume Ψ is jointly unbounded. Then the origin is globally asymptotically stable for the system*

$$s_{k+1} = T(s_k), \quad k \geq 0.$$

Proof. Since Ψ is jointly unbounded, for any $x \in \mathbb{R}_+^n$ there exists a $y \in \Psi$ such that $x \leq y$. By monotonicity we have $T^k(x) \leq T^k(y) \leq y$ for all $k \geq 0$. Hence each orbit $O(x)$ is bounded and hence has compact closure. Then by Lemma 2.2.2 we have $\omega(x) = 0$, so the origin is globally attractive. By Lemma 2.1.3 the origin is also stable, hence globally asymptotically stable. □

Also the next result is easy to see, but for future reference we put it into a lemma.

Lemma 2.2.5 *Let $T : X_+ \to X_+$ be a monotone operator. Let $x \in X_+$ and assume that there exists an $\tilde{x}$ with $\tilde{x} \geq x$, such that $O(\tilde{x})$ is bounded. Then also $O(x)$ is bounded.*

Proof. This is a direct consequence of the inequality $0 \leq x \leq \tilde{x}$. Application of T^k turns this into $0 \leq T^k(x) \leq T^k(\tilde{x})$. This proves the lemma. □

To give a resumé of this section so far, here is where we've got to: We know that global attractivity of the origin for (2.2) implies not only stability of the origin but also the no joint increase condition (2.1). The converse is wrong at least for monotone maps not induced by a matrix structure, as is outlined in Example 2.1.6. It is true for maps with a matrix structure that are moreover irreducible. The proof of Theorem 2.2.1 relies essentially

on the argument, that for primitive submatrices, each component eventually influences any other component in a $\mathcal{K}_\infty$-like way. For reducible matrix maps this particular feature does not hold. And indeed, the next example shows that without irreducibility there's no hope for global attractivity of the origin:

Example 2.2.6 *Consider the map $\Gamma_\mu : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined by $\mu = \Sigma$ and*

$$\Gamma := \begin{pmatrix} \gamma_{11} & id \\ 0 & \gamma_{22} \end{pmatrix}$$

where for $t \in \mathbb{R}_+$

$$\gamma_{11}(t) := t(1 - e^{-t})$$

and the function γ_{22} is constructed in the sequel. First note that $\gamma_{11} \in \mathcal{K}_\infty$ and $\gamma_{11}(t) < t, \forall t > 0$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a strictly decreasing sequence of positive real numbers, such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $\lim_{K\to\infty} \sum_{k=1}^K \varepsilon_k = \infty$. For $k = 1, 2, \ldots$ define

$$\gamma_{22}\left(\varepsilon_k + (1 + \sum_{j=1}^{k-1} \varepsilon_j) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)}\right) := \varepsilon_{k+1} + (1 + \sum_{j=1}^{k} \varepsilon_j) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)}$$

and observe that

$$\varepsilon_k + (1 + \sum_{j=1}^{k-1} \varepsilon_j) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)} > \varepsilon_{k+1} + (1 + \sum_{j=1}^{k} \varepsilon_j) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)},$$

since $\varepsilon_k > \varepsilon_{k+1}$ for all $k = 1, 2, \ldots$ and the map $t \mapsto t \cdot e^{-t}$ is strictly decreasing on $(1, \infty)$. Moreover we have by assumption, that

$$\varepsilon_k + (1 + \sum_{j=1}^{k-1} \varepsilon_j) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)} \xrightarrow[k\to\infty]{} 0.$$

These facts together imply that γ_{22} may be extrapolated to a $\mathcal{K}_\infty$-function, in a way such that $\gamma_{22}(t) < t$ holds for all $t > 0$. Note that by our particular construction we have $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$. Now define $s^1 \in \mathbb{R}^2_+$ by

$$s^1 := \begin{pmatrix} 1 \\ 1 + e^{-1} \end{pmatrix}$$

and, for $k = 1, 2, \ldots$, recursively define $s^{k+1} := \Gamma_\mu(s^k) \in \mathbb{R}^2_+$. By induction we verify that

$$s^{k+1} = \Gamma_\mu^k(s^1) = \begin{pmatrix} 1 + \sum_{j=1}^{k} \varepsilon_j \\ \varepsilon_{k+1} + (1 + \sum_{j=1}^{k} \varepsilon_j) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)} \end{pmatrix}.$$

By our previous considerations and assumptions we easily obtain that the second component of the sequence $\{s^k\}_{k=1}^{\infty}$ *strictly decreases and converges to zero as* k *tends to infinity. But at the same time the first component strictly increases above any given bound.*

Hence we have established that $\Gamma_\mu(s) \ngeq s \ \forall s \neq 0$ *in general does not imply global attractivity of the origin of system* (2.3)*.*

Example 2.2.7 *Note that we can turn the* 2×2*-matrix* Γ *constructed in Example 2.2.6 into null-diagonal form to conform with the structure of gain matrices in Section 3.2. Using the same notation for* γ_{ij} *as in Example 2.2.6, we just define*

$$\Gamma := \begin{pmatrix} 0 & \gamma_{11} & id & 0 \\ \gamma_{11} & 0 & 0 & id \\ 0 & 0 & 0 & \gamma_{22} \\ 0 & 0 & \gamma_{22} & 0 \end{pmatrix} \qquad \text{and} \qquad s^1 := \begin{pmatrix} 1 \\ 1 \\ 1+e^{-1} \\ 1+e^{-1} \end{pmatrix}$$

and easily verify that $\Gamma_\mu^k(s^1)$ *does not converge to* 0*.*

With Example 2.2.6 in mind the next result is somewhat astonishing: So far we focused on properties of the matrix $\Gamma \in \mathcal{G}^{n\times n}$. We completely ignored possible variations of $\mu \in \mathrm{MAF}_n^n$. There is one special case, where the structural properties of the matrix Γ do not play an important role:

Theorem 2.2.8 *Let* $\Gamma \in \mathcal{G}^{n\times n}$ *and* $\mu = \oplus$*. If* $\Gamma_\oplus(s) \ngeq s$ *for all* $s \neq 0$*, then the origin is globally asymptotically stable for*

$$s_{k+1} = \Gamma_\oplus(s_k), \quad k \geq 0.$$

The proof is kept simple by using results from Section 2.3.

Proof. Let the function σ be given by Proposition 2.3.16 or Proposition 2.3.17. For all $t \geq 0$ the function $\sigma(t) \in \Psi$ and $\sigma_i(t) \to \infty$ as $t \to \infty$, for all $i = 1, \ldots, n$. Hence Ψ is jointly unbounded. By Lemma 2.2.4 global asymptotic stability follows. □

Of course now we would like to have a result that guarantees global attractivity of the origin even if $\mu \neq \oplus$. What we find is an equivalence relation incorporating the existence of a special operator D, relating the two conditions under consideration.

Theorem 2.2.9 *Let* $\Gamma \in \mathcal{G}^{n\times n}$ *and* $\mu \in \mathrm{MAF}_n^n$*. Suppose that either*

1. Γ *is irreducible and* $\Gamma_\mu(x) \ngeq x$ *for all* $x \neq 0$ *or*

2. $\Gamma_\mu(x) \ngeq x$ *for all* $x \neq 0$ *and* $\mu = \oplus$ *or*

3. *there exists a* $D = \operatorname{diag}(id + \rho)$, $\rho \in \mathcal{K}_\infty$, *such that* $D \circ \Gamma_\mu(x) \not\geq x$ *for all* $x \neq 0$.

Then the origin is globally asymptotically stable for the system

$$s_{k+1} = \Gamma_\mu(s_k), \quad k \geq 0. \tag{2.5}$$

In case 3. there exists a $\tilde{D} = \operatorname{diag}(id + \tilde{\rho})$, $\tilde{\rho} \in \mathcal{K}_\infty$, *such that the origin is even globally asymptotically stable for the system*

$$s_{k+1} = \tilde{D} \circ \Gamma_\mu(s_k), \quad k \geq 0. \tag{2.6}$$

The next example shows that the first and second condition do not imply the third:

Example 2.2.10 *Let*

$$\Gamma = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}$$

with $\gamma(t) = t(1 - e^{-t})$ *of class* $\mathcal{K}_\infty$ *and* $\mu = \Sigma$ *or* $\mu = \oplus$. *Note that* Γ *is irreducible and conditions 1 and 2 in Theorem 2.2.9 are satisfied, but the third condition cannot be met: Assume the contrary, then also observe that*

$$(D \circ \Gamma_\mu)^2 = \begin{pmatrix} ((id + \rho) \circ \gamma)^2 & 0 \\ 0 & ((id + \rho) \circ \gamma)^2 \end{pmatrix}$$

satisfies $(D \circ \Gamma_\mu)^2(s) \not\geq s$, *by Lemma 1.5.5. By symmetry this can only be the case if* $((id + \rho) \circ \gamma)^2(t) < t$, *for all* $t > 0$. *But there is no* $\mathcal{K}_\infty$*-function* ρ *such that* $(id + \rho) \circ \gamma(t) < t$ *for all* $t > 0$*: Otherwise we had* $\rho \in \mathcal{K}_\infty$ *and* $\rho(t) < \gamma^{-1}(t) - t$ *for* $t > 0$, *which is a contradiction, since the right hand term approaches zero as* t *tends to infinity.*

Theorem 2.2.9 together with Corollary 2.1.2 immediately gives:

Corollary 2.2.11 *Let* $\Gamma \in \mathcal{G}^{n \times n}$ *and* $\mu \in \mathrm{MAF}_n^n$. *Then there exists a* $D = \operatorname{diag}(id + \rho)$, $\rho \in \mathcal{K}_\infty$, *such that* $D \circ \Gamma_\mu(x) \not\geq x$ *for all* $x \neq 0$ *if and only if there exists a* $\tilde{D} = \operatorname{diag}(id + \tilde{\rho})$, $\tilde{\rho} \in \mathcal{K}_\infty$, *such that the origin is globally asymptotically stable for the system* (2.6).

Proof of Corollary 2.2.11. One direction of the proof is directly given by Theorem 2.2.9. For the other direction we refer to Corollary 2.1.2. □

Before we prove Theorem 2.2.9 let us first prove a technical step.

Lemma 2.2.12 *Let* $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$. *Let* $D = \operatorname{diag}(id + \rho)$ *for some* $\rho \in \mathcal{K}_\infty$. *Then the following are equivalent:*

1. *For all* $x \neq 0$
$$D \circ T(x) \not\geq x;$$
2. *For all* $x \neq 0$
$$T \circ D(x) \not\geq x;$$
3. *For all* $x \neq 0$
$$D_2 \circ T \circ D_1(x) \not\geq x,$$
where D *is split into* $D = D_1 \circ D_2$ *by Lemma 1.1.4.*

Proof. The equivalence of 1. and 2. is as follows: Note that D has an inverse $D^{-1} = \operatorname{diag}((\mathrm{id}+\rho)^{-1}) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and in particular, for every $x \in \mathbb{R}^n_+$, there exists a unique y, such that $x = D(y)$ and vice versa. We obtain $D \circ T(x) \not\geq x$ if and only if $T(x) \not\geq D^{-1}(x)$. Now let $x = D(y)$ to obtain $T \circ D(y) \not\geq y$. This proves the equivalence between 1. and 2.

By Lemma 1.1.4 there exist ρ_1 and ρ_2 of class $\mathcal{K}_\infty$ such that for $D_i = \operatorname{diag}(\mathrm{id}+\rho_i)$, $i = 1, 2$, we have $D = D_1 \circ D_2$. By the same argument as in the first part of this proof we use D_1^{-1} to see that 3. is equivalent to 1. □

To keep the next proof short, we borrow again from Section 2.3.

Proof of Theorem 2.2.9. Global asymptotic stability of the origin for system (2.5) under assumptions 1 and 2 is given by Theorems 2.2.1 and 2.2.9.

Now suppose that the third assumption holds. Then by Theorem 2.3.9 there exists a path $\sigma \in \mathcal{K}^n_\infty$ satisfying $\sigma(t) \in \Omega(\Gamma_\mu)$ for all $t > 0$. Hence $\Psi(\Gamma_\mu)$ is jointly unbounded. Global asymptotic stability of the origin for system (2.5) now follows again by Lemma 2.2.4.

If we factorize $D = D_1 \circ D_2$ by Lemma 1.1.4 then we may apply the previous considerations to $D_1 \circ (D_2 \circ \Gamma_\mu)$, i.e., the origin is stable for the system (2.6) with $\tilde{D} = D_2$. □

2.3. Path construction

Consider $\Gamma \in \mathcal{G}^{n\times n}$, $\mu \in \mathrm{MAF}^n_n$, and the resulting continuous and monotone operator Γ_μ. In Section 1.5 we have found some geometrical meaning of the no joint increase condition and have defined the sets of decay. Lemma 2.2.4 states that if the set of decay Ψ is jointly unbounded, then the origin of system (2.2) is globally attractive, and by Lemma 2.1.3 it is even stable.

Now we are going to elaborate the topological properties of Ψ and Ω a bit more. A particular interest is of course if these sets are jointly unbounded. But also for future applications in Chapter 3 we are interested in paths in Ω,

parametrized by a Lipschitz continuous function σ with component functions in class $\mathcal{K}_\infty$. That is, we are looking for functions $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$, such that for $\sigma = (\sigma_1, \dots, \sigma_n)^\top$ and all $t > 0$,

$$\Gamma_\mu(\sigma(t)) \ll \sigma(t).$$

If $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is monotone and continuous and satisfies $T(x) \not\geq x$ for all $x \neq 0$, then by Proposition 1.5.3 the set $\Psi = \Psi(T)$ is radially unbounded. We always have $0 \in \Psi$. The next result shows that $x \in \Psi$ is connected to the origin by a path in Ψ.

Proposition 2.3.1 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a continuous, monotone operator satisfying $T(x) \not\geq x$ for all $x \neq 0$. Then for every $x \in \Psi$ there exists a path from 0 to x in Ψ parameterized by a nondecreasing function*

$$\sigma_x : [0, 1] \to \Psi$$

satisfying $\sigma_x(0) = 0$ and $\sigma_x(1) = x$. Moreover, σ can be chosen to be piecewise linear on every compact interval contained in $(0, 1]$. This implies that Ψ is a pathwise connected set. In addition, if T is strictly increasing, then with $x \in \Omega$ the path σ_x may be chosen to be strictly increasing and such that its image lies in $\Omega \cup \{0\}$.

In Figure 2.2 the path construction in Proposition 2.3.1 is outlined.

Proof. Given $x \in \Psi \cap S_r$ consider the sequence $\{T^k(x)\}_{k=0}^\infty$. By Lemma 1.5.8 we have $T^k(x) \in \Psi$ for all $k \geq 0$, and this sequence is nonincreasing: $T^{k+1}(x) \leq T^k(x) \leq x$ for all $k \geq 0$ by monotonicity. Also, as a trajectory starting in Ψ has a bounded orbit $O(x) \leq x$, this trajectory converges to 0 by Lemma 2.2.2. Now for every $k \geq 0$ and $\lambda \in [0, 1]$ we have

$$T^{k+1}(x) \leq (1 - \lambda)T^{k+1}(x) + \lambda T^k(x) \leq T^k(x)$$

and hence by monotonicity of T

$$T((1 - \lambda)T^{k+1}(x) + \lambda T^k(x)) \leq T^{k+1}(x) \leq (1 - \lambda)T^{k+1}(x) + \lambda T^k(x).$$

Therefore $(1-\lambda)T^{k+1}(x)+\lambda T^k(x)$ is an element of Ψ, so we get a line segment in Ψ, parametrized by λ, connecting $T^{k+1}(x)$ and $T^k(x)$.

If we let $k \to \infty$ this leads to a path in Ψ from 0 to x, parametrized, e.g., by a piecewise linear function σ_x defined this way: Let $\sigma_x(0) := 0$ and for $t \in (0, 1]$ let $k := \lfloor \frac{1}{t} \rfloor$, $\lambda := (t - \frac{1}{k+1}) \cdot (k^2 + k)$, and define

$$\sigma_s(t) := (1 - \lambda)T^k(x) + \lambda T^{k-1}(x),$$

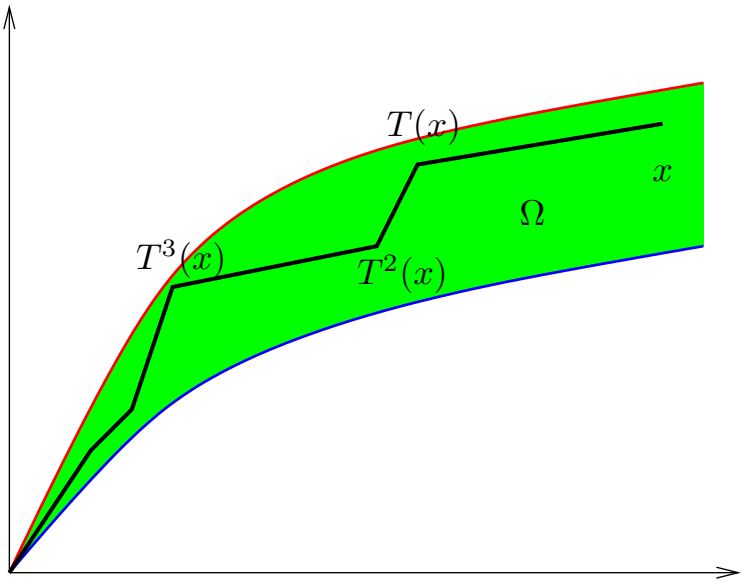


Figure 2.2.: The path in Ω, parametrized by σ_x, can be chosen to be the concatenation of line segments in Proposition 2.3.1.

where for $r \in \mathbb{R}$ the symbol $\lfloor r \rfloor$ denotes the largest integer less or equal to r.

Now let T be strictly increasing. Next we construct a path in Ω. Given $x \in \Omega$ we have $x \gg 0$ and $T^{k+1}(x) \ll T^k(x)$, by Lemma 1.5.8.2. Then also

$$T((1-\lambda)T^{k+1}(x) + \lambda T^k(x)) \ll T^{k+1}(x) \ll (1-\lambda)T^{k+1}(x) + \lambda T^k(x),$$

so $(1-\lambda)T^{k+1}(x) + \lambda T^k(x)$ is an element of Ω, which leads to a path in $\Omega \cup \{0\}$, as before.

Summarizing, this yields a strictly increasing (nondecreasing) function with values in $\Omega \cup \{0\}$ (in Ψ), if T is strictly increasing (only monotone), as can be easily verified. □

The next important question is, whether or not the set Ψ is jointly unbounded for a given operator $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Our particular attention is devoted to operators induced by $\mathcal{G}^{n\times n}$-matrices, for reasons that will become apparent in the chapter on input-to-state stability. To this end we consider the following example.

Example 2.3.2 *Let $\mu = \Sigma \in \mathrm{MAF}_2^2$ and $\Gamma \in \mathcal{G}^{2\times 2}$ be given by*

$$\Gamma = \begin{pmatrix} \gamma_{11} & id \\ 0 & \gamma_{22} \end{pmatrix},$$

where $\gamma_{11}(t) = \gamma_{22}(t) = t(1-e^{-t})$ are of class $\mathcal{K}_\infty$. Clearly $\Gamma_\mu(s) \not\geq s$ for all

$s \neq 0$ (just consider the cases $s \gg 0, s = (s_1, 0)^\top > 0$, and $s = (0, s_2)^\top > 0$ separately). What does Ω look like?

Let us consider $\Omega_1 = \{s \in \mathbb{R}^2 : s_1(1 - e^{-s_1}) + s_2 < s_1\} = \{s \in \mathbb{R}^2 : s_2 < s_1 e^{-s_1}\}$ and $\Omega_2 = \{s \in \mathbb{R}^2 : s_2(1 - e^{-s_2}) < s_2\} = \{s \in \mathbb{R}^2 : s_2 \neq 0\}$. The intersection $\Omega = \Omega_1 \cap \Omega_2$ is given by

$$\Omega = \{s \in \mathbb{R}^2 : s_2 < s_1 e^{-s_1}, s_2 \neq 0\}.$$

This set is clearly not jointly unbounded, as its vertical cross sections for large s_1 asymptotically are arbitrarily close to the s_1-axis, as depicted in Figure 2.3: The term $s_1 e^{-s_1}$ tends to zero as s_1 tends to infinity. And also, as in Example 2.2.7 by 'dimension doubling' we can turn this operator into an operator on $\mathbb{R}^4_+$ with zero diagonal.

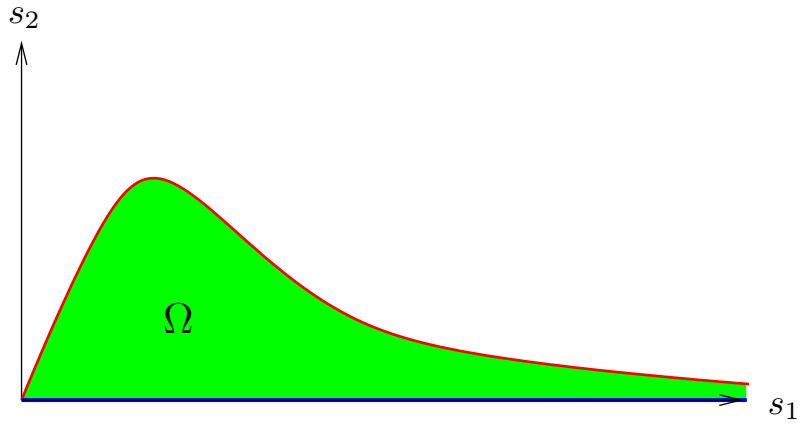


Figure 2.3.: Qualitative plot of Ω in Example 2.3.2.

So under what condition is the set Ω jointly unbounded? If we look at the linear case, i.e., any matrix $A \in \mathbb{R}^{n\times n}_+$, with $\rho(A) < 1$, then the set Ω is always jointly unbounded: By a result in [BP79, Theorem 2.1.1, p.26] there exists a nonnegative eigenvector $v \geq 0$ to the eigenvalue $\lambda = \rho(A)$, such that $Av = \lambda v < v$. If A is irreducible, then this vector is already positive and satisfies $Av = \lambda v \ll v$. If A is reducible, then for $\varepsilon > 0$ small enough, the matrix $\tilde{A} := A + \varepsilon E$, with $E = (e_{ij})$, $e_{ij} = 1$ for all i, j, still satisfies $\rho(\tilde{A}) < 1$ by continuity of the spectrum, but it is irreducible (every positive matrix is clearly primitive, and therefore irreducible). Hence by the Perron-Frobenius Theorem [BP79, Theorem 2.1.3, p.27], there exists a positive eigenvector $\tilde{v} \gg 0$ corresponding to $\rho(\tilde{A})$, such that $A\tilde{v} \ll \tilde{A}\tilde{v} = \rho(\tilde{A})\tilde{v} \ll \tilde{v}$. Now the same is true for every element $s \in \mathbb{R}_{>0} \cdot \tilde{v}$. Clearly $\mathbb{R}_{>0} \cdot \tilde{v}$ is a jointly unbounded set, $\mathbb{R}_{>0} \cdot \tilde{v} \subset \Omega(A)$, and we may view $\mathbb{R}_{>0} \cdot \tilde{v}$ as a jointly unbounded path in the set $\Omega(A)$.

It is at hand to ask, whether a jointly unbounded path, possibly of a different shape, does also exists for more general monotone operators of the form Γ_μ. The existence of such a path will later be used for the construction of Lyapunov functions in Section 3.3.4.

In this section we will construct such jointly unbounded paths using an induction argument. These paths are parameterized by functions $\sigma = (\sigma_1, \dots, \sigma_n)^\top \in \mathcal{K}_\infty^n$.

We start with an obvious result.

Lemma 2.3.3 *Let $\Gamma = (0) \in \mathcal{G}^{1\times 1}$ and $\mu \in \mathrm{MAF}_1^1$. Then $\Omega = \mathbb{R}_+ \setminus \{0\}$ and $\Psi = \mathbb{R}_+$. Hence for any $\sigma \in \mathcal{K}_\infty$ we have $\Gamma_\mu(\sigma(t)) \in \Omega$ for $t > 0$.*

The preceding lemma together with the following results on primitive and irreducible Γ forms the induction start, while the induction step finally is made in Theorem 2.3.9 for reducible Γ.

Before we start, for technical reasons we first have to study the set

$$\Psi_\infty(T) := \bigcap_{k=0}^{\infty} T^k(\Psi).$$

By definition Ψ_∞ is backward invariant: For every point in Ψ_∞ a preimage of that point is also in Ψ_∞. We are interested in the cases when this set is jointly unbounded. This is one of the major reasons why we chose $\mathcal{G}$ to be the class of function that are either of class $\mathcal{K}_\infty$ or zero, but not in $\mathcal{K} \setminus \mathcal{K}_\infty$, i.e., bounded functions.

Clearly, if T is bounded, then also Ψ_∞ is bounded. To ensure that Ψ_∞ is unbounded, we need an unboundedness requirement on T:

Definition 2.3.4 *We say that $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ satisfies the* unboundedness assumption*, if for any sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^n$ such that $\|x_k\| \to \infty$ as $k \to \infty$ we have*

$$\|T(x_k)\| \longrightarrow \infty$$

as $k \to \infty$.

Clearly, for any $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$, such that Γ has no zero column, the operator $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ satisfies the unboundedness assumption. For example, Γ has no zero column if it is irreducible.

Lemma 2.3.5 *Let $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be monotone and continuous and satisfy $T(x) \not\geq x$ for all $x \neq 0$. Assume that T satisfies the unboundedness assumption. Then $\Psi_\infty(T) \subset \Psi(T)$, $\Psi_\infty(T) \cap S_r \neq \emptyset$ for all $r \geq 0$, and $\Psi_\infty(T)$ is radially unbounded.*

Proof. By Proposition 2.3.1 the set Ψ is pathwise connected, contains 0, and is radially unbounded. It is easy to see that for any $k \geq 0$ the operator

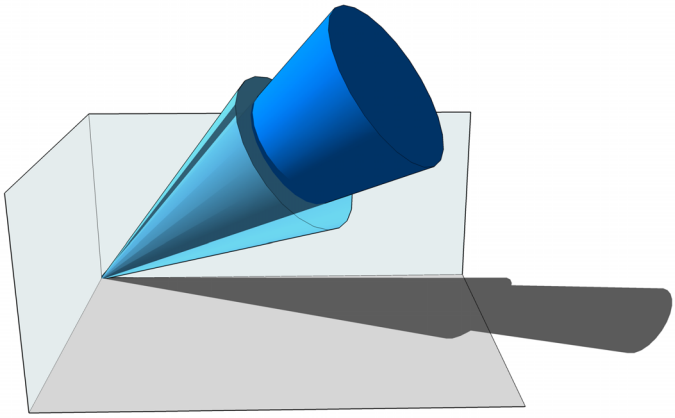


Figure 2.4.: A sketch of the set $\Psi_\infty \subset \Psi \subset \mathbb{R}^n_+$ in Lemma 2.3.5.

T^k satisfies the unboundedness assumption if T satisfies the unboundedness assumption.

For every $k \geq 0$ we have $0 \in T^k(\Psi)$, hence $0 \in \Psi_\infty \cap S_0$.

Now let $r > 0$, $k \geq 0$, and consider $M = M(k,r) = T^k(\Psi(T)) \cap S_r$. The operator T^k satisfies the unboundedness assumption and Ψ is radially unbounded, so there exists an $x \in \Psi(T)$ such that $\|T^k(x)\|_1 \geq r$. Since $\Psi(T)$ is pathwise connected there exists a path connecting 0 and x in $\Psi(T)$. By continuity of T^k this leads to a path connecting 0 and $T^k(x)$ in $T^k(\Psi(T))$. Clearly this path intersects M, so M is nonempty. It is also clear that M is bounded. Now we show that M is closed: Let $\{x^l\}_{l\in\mathbb{N}}$ be a Cauchy sequence in M. Since we are in Euclidean space, we have $x^l \to x$ for some $x \in \mathbb{R}^n_+$ as $l \to \infty$. Since $\{x^l\}_{l\in\mathbb{N}} \subset T^k(\Psi(T))$ there exist $y^l \in \Psi(T)$ such that $x^l = T^k(y^l)$. Assume the sequence $\{y^l\}$ was unbounded. Then by the unboundedness assumption also $\{x^l\}$ had to be unbounded, in contradiction that x^l converges to x. Hence $\{y^l\}_{l\in\mathbb{N}}$ is a bounded sequence in $\Psi(T) \subset \mathbb{R}^n_+$, so it has a convergent subsequence, which for notational simplicity we again denote by $\{y^l\}$. Denote its limit by y, which is an element of $\Psi(T)$, since $\Psi(T)$ is closed. So by continuity we conclude $x = T^k(y)$, proving that M is closed. Together with boundedness we have that M is compact for any choice of k and r.

By Lemma 1.5.8 the sets $T^k(\Psi)$ are nested. So the sets $M^k_r = M(k,r)$, $k \geq 0$, are nested and compact for fixed $r > 0$.

A decreasing family of compacts sets has nonempty intersection (see e.g. [Heu00a, p.230]), so for any $r > 0$ the set $M_r = \bigcap_k M^k_r$ is nonempty. This proves the claim. □

Proposition 2.3.6 *Let $\Gamma \in \mathcal{G}^{n\times n}$ be primitive and $\mu \in \mathrm{MAF}_n^n$. Suppose $\Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. Then there exists a function $\sigma \in \mathcal{K}_\infty^n$ such that*

$$\sigma(t) \in \Omega(\Gamma_\mu), \quad \textit{i.e.,} \quad \Gamma_\mu(\sigma(t)) \ll \sigma(t)$$

for all $t > 0$. And this function σ can be chosen to be piecewise linear on every compact interval contained in $(0, \infty)$.

Proof. By Proposition 1.5.3 the set Ψ is radially unbounded. By primitivity of Γ there exists a k_0 such that $\pi_i \circ \Gamma_\mu^k \circ \iota_j \in \mathcal{K}_\infty$ for any $i, j \in \{1, \ldots, n\}$ and $k \geq k_0$. Hence the sets $\Gamma_\mu^k(\Psi)$ are jointly unbounded for $k \geq k_0$. Together with Lemma 2.3.5 we have that Ψ_∞ is nonempty, jointly unbounded, and closed. By Lemma 1.3.2 the operator Γ_μ is eventually strongly monotone and hence by Lemma 1.5.8 $\Psi_\infty \setminus \{0\} \subset \Omega$.

Pick $s \in \Psi_\infty$, then by Lemma 1.5.8 $\Gamma^k(s) \in \Psi_\infty$ for all $k \geq 0$. We start with $\sigma_s : [0, 1] \to \Omega \cup \{0\}$ taken from Proposition 2.3.1.

Since we may always pick a preimage in Ψ_∞ we define an ascending sequence by $s^0 = s$, $s^{k+1} \in \Gamma_\mu^{-1}(s^k) \subset \Psi_\infty$. Next we extend σ_s to a function σ on $\mathbb{R}_+$ by defining $\sigma|_{[0,1]} = \sigma_s$ and for $t \in]1, \infty[$ by

$$\sigma(t) = (1 - t + \lfloor t \rfloor)s^{\lfloor t \rfloor - 1} + (t - \lfloor t \rfloor)s^{\lfloor t \rfloor} \in \Omega, \tag{2.7}$$

where $\lfloor t \rfloor$ denotes the greatest integer less or equal to t. We have $\sigma(t) \in \Omega$ for $t > 1$ since Γ_μ is strictly increasing and for $x \in \Omega$ we have $\Gamma_\mu^2(x) \ll \Gamma_\mu(x) \ll x$ and hence for $\lambda \in]0, 1[$ that

$$\Gamma_\mu(x) \ll (1 - \lambda)\Gamma_\mu(x) + \lambda\Gamma_\mu(x) \ll x$$

and hence

$$\Gamma_\mu^2(x) \ll \Gamma_\mu((1 - \lambda)\Gamma_\mu(x) + \lambda\Gamma_\mu(x)) \ll \Gamma_\mu(x) \ll (1 - \lambda)\Gamma_\mu(x) + \lambda\Gamma_\mu(x).$$

The function σ is continuous by definition. To check that σ is strictly increasing, it suffices to note that σ_s is strictly increasing by Proposition 2.3.1 and to note that the sequence $\{\Gamma_\mu^{-k}(s)\}_{k\geq 0} \subset \Psi_\infty \subset \Omega \cup \{0\}$ is strictly increasing in every component, for Γ_μ is strictly increasing.

It remains to prove unboundedness of the component functions. First assume σ is bounded. Since σ is nondecreasing, there must exist a limit point

$$s^* := \lim_{k\to\infty} \sigma(k) = \lim_{k\to\infty} \Gamma_\mu(\sigma(k + 1)) = \Gamma_\mu(\lim_{k\to\infty} \sigma(k + 1)) = \Gamma_\mu(s^*),$$

but since $\sigma(1) \gg 0$ and σ is nondecreasing, and hence $s^* \gg 0$, this contradicts $\Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. So there exists at least one unbounded component σ_i

of σ. From irreducibility (primitive matrices are also irreducible) we deduce that there exists another unbounded component (each component eventually affects every other component): There exists a $j \neq i$, such that $\gamma_{ji} \in \mathcal{K}_\infty$. Hence if $\sigma_i(t) \to \infty$ as $t \to \infty$, so does $\sigma_j(t) > \pi_j \circ \Gamma_\mu(\sigma(t)) \geq \gamma_{ji}(\sigma_i(t))$, since $\Gamma_\mu(\sigma(t)) \ll \sigma(t)$ for all $t > 0$ by (2.7). Inductively and by irreducibility of Γ we obtain that all components are unbounded. This proves the proposition. □

Proposition 2.3.7 *Let $\Gamma \in \mathcal{G}^{n\times n}$ be irreducible and $\mu \in \mathrm{MAF}_n^n$. Suppose there exists a $D = \mathrm{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty$, $\rho > id$, such that $D \circ \Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. Then there exists a function $\sigma \in \mathcal{K}_\infty^n$ such that*

$$\sigma(t) \in \Omega(\Gamma_\mu)$$

for all $t > 0$. Moreover, this function σ can be chosen to be piecewise linear on every compact interval contained in $(0, \infty)$.

REMARK. After the submission of this thesis, but before its publication, the author has been able to prove the existence of a diagonal operator $D = \mathrm{diag}(\rho)$, $\rho \in \mathcal{K}_\infty$, $\rho > \mathrm{id}$ also for general $\mu \in \mathrm{MAF}_n^n$ for irreducible $\Gamma \in \mathcal{G}^{n\times n}$ satisfying $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$. The assumption of the existence of D could therefore be dropped in Proposition 2.3.7, only $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$ is needed.

However, this result is not proved or further embedded into the structure of the document. But see Proposition 2.3.16, where the operator D is constructed for the case $\mu = \oplus$.

Proof. The matrix Γ has no zero rows, hence Γ_μ is strictly increasing and so is $\Gamma_\mu \circ D$. Let $T := \Gamma_\mu \circ D$. By Lemma 2.2.12 we have $T(s) \not\geq s$ for all $s \neq 0$.

By Lemma 2.3.5 the set $\Psi_\infty(T)$ is radially unbounded, so we may pick an unbounded sequence $\{r^k\} \subset \Psi_\infty(T)$, such that for some index i we have $r_i^k \to \infty$ as $k \to \infty$.

Now Γ is irreducible, and hence T satisfies the unboundedness assumption. It follows that also T^l satisfies the unboundedness assumption for all $l \geq 0$.

By irreducibility of Γ for any index j there exists an $l = l(j)$ such that

$$\pi_j \circ T^l(r^k) \to \infty$$

as $k \to \infty$. Also we have $\{T^l(r^k)\}_{k\geq 0} \subset \Psi_\infty(T)$ and for every $k \geq 0$, $T^l(r^k) \leq r^k$. It follows that $\{r^k\}$ is jointly unbounded, proving that $\Psi_\infty(T)$ is jointly unbounded.

By Lemma 1.5.9 we have $\Psi_\infty(T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\} \subset \Psi(T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\} \subset \Omega(\Gamma_\mu)$. So since $\Psi_\infty(T)$ is jointly unbounded, we may pick a point $s^0 \in \Psi_\infty(T) \cap \{s \in \mathbb{R}^n_+ : s \gg 0\}$.

Now define a sequence $\{s^k\}_{k\geq 0}$ by

$$s^{k+1} \in T^{-1}(s^k) \cap \Psi_\infty(T)$$

for $k \geq 0$. This is possible, since $\Psi_\infty(T)$ is backward invariant under T.

The so defined sequence $\{s^k\}$ satisfies $s^k < s^{k+1}$ by definition. We claim that it is also jointly unbounded: To this end assume that it is bounded. Then by monotonicity there exists a limit $s^* = \lim_{k\to\infty} s^k$. By continuity of T and since $s^k = T(s^{k+1})$ we have

$$s^* = \lim_{k\to\infty} s^k = \lim_{k\to\infty} T(s^{k+1}) = T\left(\lim_{k\to\infty} s^{k+1}\right) = T(s^*)$$

contradicting $T(x) \not\geq x$ for all $x \neq 0$. Hence the sequence $\{s^k\}$ must be radially unbounded. By the same argument as earlier for the sequence $\{r^k\}$ we see that $\{s^k\}$ is indeed jointly unbounded.

Now the sequence $\{s^k\}$ lives in $\Omega(\Gamma_\mu)$, but it may not be strictly increasing, as we only know $s^k < s^{k+1}$ for all $k \geq 0$. We define a strictly increasing sequence $\{\tilde{s}^k\}$ as follows: By Lemma 1.1.4 for any $k \geq 0$ we may factorize $D = D_1^{(k)} \circ D_2^{(k)}$ in such a way that $D_2^{(k)}(s) \ll D_2^{(k+1)}(s)$ for all $k \geq 0$ and all $s \gg 0$. Using this factorization we define

$$\tilde{s}^k := D_2^{(k)}(s^k)$$

for all $k \geq 0$. By the definition of $D_2^{(k)}$, this sequence is clearly strictly increasing and inherits from $\{s^k\}$ the joint unboundedness.

We claim that $\{\tilde{s}^k\} \subset \Omega(\Gamma_\mu)$: This follows from

$$\tilde{s}^k \gg s^k \geq \Gamma_\mu \circ D(s^k) = \Gamma_\mu \circ D_1^{(k)} \circ D_2^{(k)}(s^k) = \Gamma_\mu \circ D_1^{(k)}(\tilde{s}^k) \gg \Gamma_\mu(\tilde{s}^k).$$

Now we prove that for $\lambda \in (0,1)$ we have $(1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_\mu)$. Clearly

$$\tilde{s}^k \ll (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \ll \tilde{s}^{k+1}$$

and application of the strictly increasing operator Γ_μ yields

$$\begin{aligned}\Gamma_\mu((1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1}) &\ll \Gamma_\mu(\tilde{s}^{k+1}) \\ &= \Gamma_\mu \circ D_2^{(k+1)}(s^{k+1}) \ll \Gamma_\mu \circ D_1^{(k+1)} \circ D_2^{(k+1)}(s^{k+1}) \\ &= s^k \ll \tilde{s}^k \ll (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1}.\end{aligned}$$

Hence $(1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_\mu)$.

Now we may define σ as a parametrization of the linear interpolation of the points $\{\tilde{s}^k\}_{k\geq 0}$ in the unbounded direction as in the proof of Proposition 2.3.6, and utilize $\sigma_{\tilde{s}^0}$ from Proposition 2.3.1 for the other direction.

Clearly this function σ has component functions of class $\mathcal{K}_\infty$ and is piecewise linear on every compact interval contained in $(0,\infty)$. □

The preceding proof would have been easier, if Γ_μ being strictly increasing together with $s^0 \gg 0$ implied $s^k \ll s^{k+1}$ for $k \geq 0$. But this is not true, as the following example shows:

Example 2.3.8 *In this example we show that it is possible that trajectories starting in the boundary of Ω always remain therein. Let*

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix}$$

with $\gamma_{12}(t) = \sqrt{t}$ and $\gamma_{21}(t) = \frac{1}{2}t^2$ for $t \geq 0$. Let $\mu = \Sigma$ or $\mu = \oplus$. Then for $s \in \mathbb{R}^2_+$ we have

$$\Gamma_\mu(s) = \begin{pmatrix} \sqrt{s_2} \\ \frac{1}{2}s_1^2 \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad \text{if } s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \neq 0.$$

Now let $t > 0$. Then for $s = (\sqrt{t}, t)^\top$ we find

$$0 \ll \Gamma_\mu(s) = \begin{pmatrix} \sqrt{t} \\ \frac{1}{2}t \end{pmatrix} < \begin{pmatrix} \sqrt{t} \\ t \end{pmatrix} = s,$$

but $\Gamma_\mu(s) \not\ll s$. Also

$$\Gamma_\mu^2(s) = \begin{pmatrix} \sqrt{\frac{1}{2}t} \\ \frac{1}{2}t \end{pmatrix}$$

is of the same form as s, hence the trajectory starting in s will always remain in the boundary of Ω.

The example shows that $\Psi_\infty(\Gamma_\mu)$ does not 'shrink' to be contained in $\Omega(\Gamma_\mu)$ as in the primitive case. Therefore the operator D is an essential ingredient. Note that in contrast to former occurrences of diagonal operators D in this work, this time D is only of the form $D = \operatorname{diag}(\rho)$ instead of $D = \operatorname{diag}(\mathrm{id} + \rho)$ for some $\rho \in \mathcal{K}_\infty$. In Proposition 2.3.16 we are going to show that if $\mu = \oplus$, then a diagonal operator $D = \operatorname{diag}(\rho)$ with $\rho \in \mathcal{K}_\infty$ always exists for irreducible Γ, if $\Gamma_\oplus(s) \not\geq s$ for all $s \neq 0$.

Now we state the main result of this section, which assures the existence of a path in the decay set $\Omega(\Gamma_\mu)$, parametrized by $\mathcal{K}_\infty$ functions, for general

matrix induced operators Γ_μ. This time, instead of structural assumptions on Γ, we make a stronger assumption on the operator D in addition to the no joint increase condition.

Theorem 2.3.9 *Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. Suppose there exists a $D = \mathrm{diag}(id + \rho)$ for some $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. Then there exists a function $\sigma \in \mathcal{K}_\infty^n$ such that*

$$\sigma(t) \in \Omega(\Gamma_\mu)$$

for all $t > 0$, and σ can be chosen to be piecewise linear on every compact interval contained in $(0, \infty)$.

The constructive arguments in the previous path constructions for primitive and irreducible matrices Γ always resulted in a path that is piecewise linear on every compact interval contained in $(0, \infty)$. This time we have to apply one more technical argument to ensure this property in the case that Γ is reducible. This argument is put into the following Lemma.

Lemma 2.3.10 *Let $\tau \in \mathcal{K}_\infty$. Then there exists a $\mathcal{K}_\infty$ function $\hat{\tau}$ such that $\hat{\tau} \leq \tau$ and $\hat{\tau}$ is piecewise linear on every compact interval contained in $(0, \infty)$.*

Proof. We cover $(0, \infty)$ by countably many compact intervals in such a way, that two consecutive intervals only intersect in one point: For $n \in \mathbb{Z}$ Let

$$I_n := \begin{cases} [-\frac{1}{n-1}, -\frac{1}{n}] & \text{if } n < 0, \\ [n+1, n+2] & \text{if } n \geq 0. \end{cases}$$

The family $\{I_n\}_{n\in\mathbb{Z}}$ will do the job. We now construct a function $\hat{\tau}$ on each set $I = [a, b] \in \{I_n\}_{n\in\mathbb{Z}}$ individually, so that at the end, $\hat{\tau}$, seen as a function $\mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$, is piecewise linear on any compact interval in $(0, \infty)$, and satisfies $\hat{\tau}(t) < \tau(t)$ for $t > 0$. We start by defining $\hat{\tau}(0) = 0$.

On every interval $I = [a, b] \in \{I_n\}_{n\in\mathbb{Z}}$ the function τ is strictly increasing and we have $\min_{t\in I} \tau(t) > 0$. So on I we construct $\hat{\tau}$ as indicated in Figure 2.5:

For some $\varepsilon \in (0, 1)$ to be specified in a moment, we draw a line segment from $(a, \frac{1}{2}\tau(a))$ to $(b, \tau(a))$ and another one from $(\varepsilon a+(1-\varepsilon)b, 0)$ to $(b, \frac{1}{2}\tau(b))$. We have to ensure that especially the second line segment lies below the graph of τ, which we do in a moment, by choosing ε in the right way.

Then we take $\hat{\tau}$ as the maximum of these two linear functions, i.e.,

$$\hat{\tau}(t) = \max\left\{\frac{1}{2}\frac{\tau(a)}{(b-a)} \cdot (t-a) + \frac{1}{2}\tau(a), \frac{1}{2}\frac{\tau(b)}{\varepsilon(b-a)} \cdot (t-b) + \frac{1}{2}\tau(b)\right\}$$

for $t \in I$.

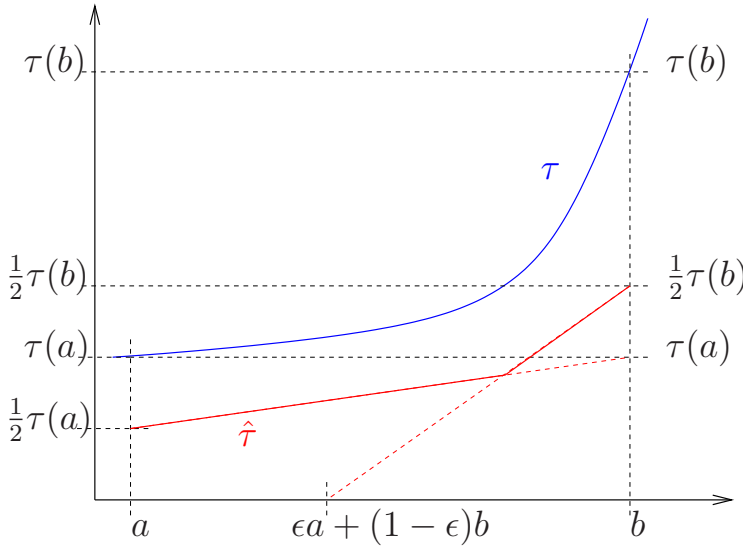


Figure 2.5.: How to bound a $\mathcal{K}_\infty$ function τ from below with a piecewise linear $\mathcal{K}_\infty$ function $\hat{\tau}$.

To ensure that the graph of $\hat{\tau}$ lies below the graph of τ, we have to choose ε sufficiently large. There are two cases:

1. $\tau(a) \geq \frac{1}{2}\tau(b)$: In this case any choice of $\varepsilon \in (0,1)$ gives two lines segments that lie entirely below the graph of τ on the interval I.

2. $\tau(a) < \frac{1}{2}\tau(b) < \tau(b)$: By the Intermediate Value Theorem, well known from elementary calculus, there exists a point $\xi \in (a,b)$, such that $\tau(\xi) = \frac{1}{2}\tau(b)$. Since τ is strictly increasing, this ξ is unique.

   The first line segment has no chance of intersecting the graph of τ, but the second one might potentially intersect the graph of τ somewhere on the interval (a,ξ). To prevent that, we chose, e.g., ε larger than $(\xi - a)/(b-a)$ so that $\varepsilon a + (1+\varepsilon)b \geq \xi$. In fact, a smaller choice for ε would be possible as well, as is indicated in Figure 2.5, so that the intersection point of the line segments is less or equal to ξ. □

Now we proceed with the proof for the construction of the path σ in $\Omega(\Gamma_\mu)$ for general matrices Γ.

Proof of Theorem 2.3.9. We may assume that if Γ is reducible, it already is of upper block diagonal form, using formula (1.5).

By Lemma 1.1.4 we may assume that there exists a $D = \operatorname{diag}(\mathrm{id} + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ D \circ \Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. We construct σ by induction over the number of irreducible and 1×1-zero blocks on the

diagonal of Γ. The argument is as follows.

Induction start. If Γ is irreducible or a 1×1-zero-block, the claim follows from Lemma 2.3.3 or Proposition 2.3.7. We may assume that the function σ given by these results is piecewise linear on compact intervals contained in $(0, \infty)$.

Induction step. A reducible matrix in upper triangular block form with a total of $k+1$ irreducible and zero 1×1-blocks B_l, $l = 1, \dots, k+1$, on the diagonal can be considered as a block matrix with only two diagonal blocks, the first of which possibly reducible (if $k > 1$) and the other one zero/irreducible. So may assume that Γ is reducible into

$$\Gamma = \left(\begin{array}{ccc|c} B_1 & * & * & * \\ 0 & \ddots & * & * \\ 0 & 0 & B_k & * \\ \hline 0 & 0 & 0 & B_{k+1} \end{array}\right) = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}$$

for some $\Gamma_{11} \in \mathcal{G}^{n_1 \times n_1}$, $\Gamma_{12} \in \mathcal{G}^{n_1 \times n_2}$, and $\Gamma_{22} \in \mathcal{G}^{n_2 \times n_2}$ for some $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, so that $\Gamma_{22} = B_{k+1}$ is irreducible or a 1×1-zero block. We may further assume that we have (with D adapted according to the dimensions)

$$D^2 \circ (\Gamma_{ii})_{\mu_i}(x) \not\geqslant x$$

for all $x \in \mathbb{R}_+^{n_i}$, $i = 1, 2$, with $\mu = (\mu_1, \mu_2)$.

Induction hypothesis: There exists a function $\sigma_1 \in \mathcal{K}_\infty^{n_1}$ such that

$$D \circ (\Gamma_{11})_{\mu_1}(\sigma_1(t)) \ll \sigma_1(t) \tag{2.8}$$

for all $t > 0$. Now, since Γ_{22} is irreducible or a 1×1-zero-block, by Lemma 2.3.3 or Proposition 2.3.7, there also exists a function $\sigma_2 \in \mathcal{K}_\infty^{n_2}$ satisfying

$$D \circ (\Gamma_{22})_{\mu_2}(\sigma_2(t)) \ll \sigma_2(t) \tag{2.9}$$

for all $t > 0$. There is no restriction in assuming that both σ_1 and σ_2 are piecewise linear on compact intervals in $(0, \infty)$. By Lemma 1.1.4 we may again alter D such that (2.8) and (2.9) can be rewritten as

$$D^2 \circ (\Gamma_{ii})_{\mu_i}(\sigma_i(t)) \ll \sigma_i(t)$$

or equivalently

$$D \circ (\Gamma_{ii})_{\mu_i}(\sigma_i(t)) \ll D^{-1}(\sigma_i(t))$$

for all $t > 0$. By Lemma 1.1.5 D^{-1} is of the form $\operatorname{diag}(\mathrm{id} - \tilde{\rho})$ for some $\tilde{\rho} \in \mathcal{K}_\infty$.[1] Write $R = \operatorname{diag}(\tilde{\rho})$. Define a function $\tau : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$,

$$\tau(t) := \sup\{r \in \mathbb{R}_+ : D \circ (\Gamma_{12})_\mu(\sigma_2(r)) \leq R(\sigma_1(t))\}.$$

[1] This is exactly the point were it is essential that we have D of the form $\operatorname{diag}(\mathrm{id} + \rho)$ instead of $\operatorname{diag}(\rho)$.

If $\Gamma_{12} \neq 0$ then this function is strictly increasing and as $t \to \infty$ we have $\tau(t) \to \infty$. In addition τ satisfies $\tau(0) = 0$ and $\tau(t) > 0$ for all $t > 0$. Clearly τ is continuous and therefore of class $\mathcal{K}_\infty$. If $\Gamma_{12} = 0$ we can bound τ from below by a function of class $\mathcal{K}_\infty$. For simplicity we denote this function again by τ.

Now we apply Lemma 2.3.10 to bound τ from below with a function $\hat{\tau}$ of class $\mathcal{K}_\infty$ that is piecewise linear on any compact interval in $(0, \infty)$.

Hence for all $t \geq 0$ we have

$$D \circ (\Gamma_{12})_\mu(\sigma_2(\hat{\tau}(t))) \leq R(\sigma_1(t)).$$

Now consider

$$\begin{aligned}
\pi_{I_1} \circ D \circ \Gamma_\mu((\sigma_1(t), \sigma_2 \circ \hat{\tau}(t))^\top) &\leq D \circ (\Gamma_{11})_\mu(\sigma_1(t)) \\
&\quad + D \circ (\Gamma_{12})_\mu \circ \sigma_2 \circ \hat{\tau}(t) \\
&\ll D^{-1}(\sigma_1(t)) + D \circ (\Gamma_{12})_\mu \circ \sigma_2 \circ \hat{\tau}(t) \\
&= \sigma_1(t) - R(\sigma_1(t)) \\
&\quad + D \circ (\Gamma_{12})_\mu \circ \sigma_2 \circ \hat{\tau}(t) \\
&\leq \sigma_1(t).
\end{aligned}$$

Hence we have established

$$D \circ \Gamma_\mu \left(\begin{pmatrix} \sigma_1(t) \\ \sigma_2 \circ \hat{\tau}(t) \end{pmatrix} \right) \ll \begin{pmatrix} \sigma_1(t) \\ \sigma_2 \circ \hat{\tau}(t) \end{pmatrix} =: \sigma(t),$$

implying that $\sigma(t) \in \Omega(D \circ \Gamma_\mu) \subset \Omega(\Gamma_\mu)$ for all $t > 0$. By construction σ is piecewise linear on every compact interval contained in $(0, \infty)$. This completes the induction step. □

A natural thing to ask now would be, if the special form of the operator $D = \operatorname{diag}(\mathrm{id} + \rho)$, $\rho \in \mathcal{K}_\infty$, in the assumptions of Theorem 2.3.9 is indeed necessary, or if this result is just not as sharp as it could be.

The answer lies is Example 2.3.2: There a reducible matrix Γ is given, so that for $\mu = \Sigma$ the condition

$$\Gamma_\mu(x) \not\geq x \text{ for all } x \neq 0$$

holds, but a function $\sigma \in \mathcal{K}_\infty^2$ such that

$$\Gamma_\mu(\sigma(t)) \ll \sigma(t) \text{ for } t > 0$$

clearly cannot exist, because Ω is bounded in one coordinate direction.

Now we could ask, if a diagonal operator of the form $D = \operatorname{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty$, $\rho > \mathrm{id}$, would not have been sufficient. The answer is again no, as further elaboration of the very same example shows:

Example 2.3.11 *We reconsider Example 2.3.2. There we had* $\mu = \Sigma$ *and* $\Gamma \in \mathcal{G}^{2\times 2}$ *given by*

$$\Gamma = \begin{pmatrix} \gamma_{11} & id \\ 0 & \gamma_{22} \end{pmatrix},$$

where $\gamma(t) = \gamma_{11}(t) = \gamma_{22}(t) = t(1 - e^{-t})$ *are of class* $\mathcal{K}_\infty$*. So* Γ *is reducible.*

Now let $\rho \in \mathcal{K}_\infty$ *be given as* $\rho(t) = t(1+e^{-t})$*. Then* $\rho\circ\gamma(t) = t(1-e^{-2t}) < t$ *for all* $t > 0$ *and clearly* ρ *is not of the form* $id + \mathcal{K}_\infty$*, but satisfies* $\rho > id$*. Write* $D = \operatorname{diag}(\rho)$*. Using the same argument as in Example 2.3.2 we see that* $\Gamma_\mu \circ D(x) \not\geq x$ *for all* $x \neq 0$*, which is equivalent to* $D \circ \Gamma_\mu(x) \not\geq x$ *for all* $x \neq 0$*, by Lemma 2.2.12.*

Remark 2.3.12 *Note that the functions* σ *constructed in the proofs of Propositions 2.3.6, 2.3.7, and Theorem 2.3.9 are locally Lipschitz continuous on* $]0, \infty[$ *by construction.*

Summarizing, we established that for operators Γ_μ induced by $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$ the decay sets Ω are not only radially unbounded if $\Gamma_\mu(s) \not\geq s$ for all $s \neq 0$, but also there exists a locally Lipschitz continuous path $\sigma : \mathbb{R}_+ \to \Omega$ with component functions of class $\mathcal{K}_\infty$ if the stronger condition holds, that there exists a $D = \operatorname{diag}(\mathrm{id} + \rho)$ for some $\rho \in \mathcal{K}_\infty$, such that $D(\Gamma_\mu(s)) \not\geq s$ for all $s \neq 0$.

Example 2.3.13 *As follows from Example 2.1.6, for general monotone operators* $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ *the set* Ψ *may not be jointly unbounded if we only have* $T(s) \not\geq s$ *for all* $s \neq 0$*. Let us reconsider Example 2.1.6. For the operator* T *we can also write down a* $D = (1+\alpha)id_{\mathbb{R}_+^2}$*, where* $0 < \alpha < 1/\lambda - 1$*, satisfying* $(D \circ T)(s) \not\geq s$ *for all* $s \neq 0$*. (Again, just consider the cases* ($s_1 = 0$, $s_2 > 0$) *and* $s_2 = 0$ *separately to see this.) Geometrically we have*

$$\begin{aligned} \Omega_1 &= \{s \in \mathbb{R}_+^2 : \lambda s_1 + s_1^2 s_2 + \mu s_2 < s_1\} \\ &= \left\{ s \in \mathbb{R}_+^2 : s_1 > 0,\ s_2 < \frac{(1-\lambda)s_1}{s_1^2 + \mu} \right\} \\ \Omega_2 &= \{s \in \mathbb{R}_+^2 : \lambda s_2 < s_2\} = \{s \in \mathbb{R}_+^2 : s_2 \neq 0\}, \end{aligned}$$

where Ω_i *has been defined in Section 1.5 and denotes the set where the* i*th component decreases with respect to the map* T*, i.e,*

$$\Omega_i = \{s \in \mathbb{R}_+^2 : (T(s))_i < s_i\}, \quad i = 1, 2.$$

By definition the set $\Psi = \Psi(T)$ *consists all those points* $s \in \mathbb{R}_+^2$ *satisfying* $T(s) \leq s$*, and in this example we have* $\Psi = \overline{\Omega_1 \cap \Omega_2}$*. The picture in Figure 2.6 is drawn for* $\lambda = 1/2$ *and* $\mu = 1/16$*. We see that for general monotone operators* $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ *a construction of an jointly unbounded path in* Ψ_∞

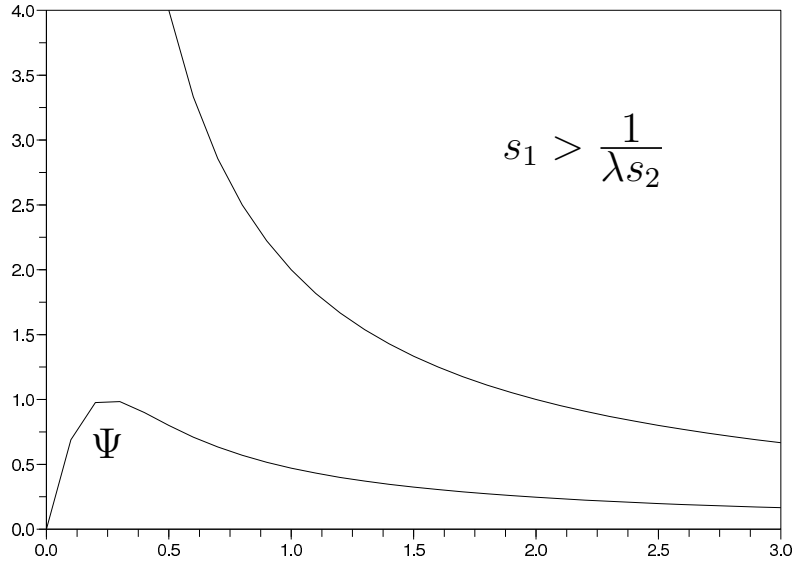


Figure 2.6.: Attracting and repelling sets in Example 2.3.13

or Ω is not possible, as the decay sets Ω are not jointly unbounded in the first place. Nevertheless, for maps on two dimensional space, things are a bit simpler, as we are going to see in the next section.

If $\mu = \oplus$ then we can omit the scaling operator D, as we will see in the following. In fact, a diagonal operator $D = \mathrm{diag}(\rho)$, $\rho \in \mathcal{K}_\infty$, $\rho > \mathrm{id}$, always exists, such that $\Gamma_\mu(D(s)) \not\geq s$ for $s \neq 0$. The next statements also relate our results to the existing literature, see the discussion in Section 2.7.

It is interesting how the no joint increase condition relates to the well known cycle condition: A function $\gamma \in \mathcal{G}$ is a *contraction*, if $\gamma(t) < t$ for all $t > 0$. We abbreviate this by $\gamma < \mathrm{id}$. A k-cycle in a matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n \in \mathcal{G}^{n\times n}$ is a sequence of $\mathcal{K}_\infty$ functions

$$(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \ldots, \gamma_{i_{k-1} i_k})$$

of length k, i.e., $\gamma_{i_l, i_{l+1}} \in \mathcal{K}_\infty$ for $l = 0, \ldots, k-1$ and the indices $i_0, \ldots, i_k \in \{1, \ldots, n\}$ are such that $i_0 = i_k$. The k-cycle is *subordinated*, if $i_0 \geq i_l$ for all $l \in \{1, \ldots, k\}$. If $i_l \neq i_j$ for all $l \neq j$ other than for the required $i_0 = i_k$, then the k-cycle is called *minimal*.

The *cycle condition* for a matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n \in \mathcal{G}^{n\times n}$ is that all k-cycles of Γ are contractions, i.e.,

$$\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_{k-1} i_k} < \mathrm{id},$$

for all $i_0, \ldots, i_k \in \{1, \ldots, n\}$ with $i_0 = i_k$ and $k \leq n$. This allows us to formulate the following equivalence:

Lemma 2.3.14 *Let $\Gamma \in \mathcal{G}^{n\times n}$. Then Γ satisfies the cycle condition if and only if $\Gamma_\oplus(x) \not\geq x$ for all $x > 0$.*

Proof. The no joint increase condition implies the cycle condition: We argue by contradiction. Suppose there is a k-cycle such that for a $t \in \mathbb{R}_+$ we have $\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_{k-1} i_k}(t) \geq t$ with $i_0 = i_k$. Then $\pi_{i_0}(\Gamma^k_\oplus(t \cdot e_{i_0})) \geq \gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_{k-1} i_k}(t) \geq t$ and hence $\Gamma^k_\oplus(t \cdot e_{i_0}) \geq t \cdot e_{i_0}$. But this contradicts $\Gamma^k_\oplus(x) \not\geq x$ for all $x \neq 0$, which is necessary for the no joint increase condition to hold, see Lemma 1.5.5. Hence the no joint increase condition cannot hold. This proves one implication.

To prove the converse we argue again by contradiction. Assume the cycle condition is true while there exists an $x > 0$, such that $\Gamma_\oplus(x) \geq x$. Choose i such that x_i is maximal, i.e., $x_i = \max_j x_j$. By monotonicity of $\Gamma_\oplus$ we have $x \leq \Gamma^k_\oplus(x)$ for all $k \geq 1$ and hence

$$\begin{aligned} 0 < x_i &\leq \pi_i \circ \Gamma^k_\oplus(x) \\ &= \max_{l_1,\ldots,l_{k-1},j} \gamma_{il_1} \circ \ldots \circ \gamma_{l_{k-1}j}(x_j) \\ &\leq \max_{l_1,\ldots,l_{k-1},j} \gamma_{il_1} \circ \ldots \circ \gamma_{l_{k-1}j}(x_i). \end{aligned} \tag{2.10}$$

Now the number of functions γ_{lm} in each term in the maximization in (2.10) that are not part of a cycle is bounded by $2(n-1)$. To see this, note that a loop-free cycle of maximal length in a graph with n vertices consists of at most $n-1$ edges. This implies that the composition $\gamma_{il_1} \circ \ldots \circ \gamma_{l_{k-1}j}$ in (2.10) can be written as

$$\underbrace{\gamma_{il_1} \circ \ldots \circ \gamma_{l_{\lambda-1}l_\lambda}}_{\text{without loops}} \circ \underbrace{\gamma_{l_\lambda l_{\lambda+1}} \circ \ldots \circ \gamma_{l_{\nu-1}l_\nu}}_{\text{only loops}} \circ \underbrace{\gamma_{l_\nu l_{\nu+1}} \circ \ldots \circ \gamma_{l_{k-1}j}}_{\text{without loops}} \tag{2.11}$$

where the first and the last part are possibly of zero length, but each at most of length $n-1$ each.

As k tends to infinity, the number of cycles in (2.11), respectively in the right hand side (2.10) increases. Each such cycle is less than the identity, hence for $k \to \infty$ the right hand side (2.10) tends to zero, contradicting $x_i > 0$. □

Remark 2.3.15 *If the functions γ_{ij} are all linear, then $\Gamma_\oplus$ is a max linear operator, see* [Lur05]. *Note that this is not the same as a max-plus linear operator. The cycle condition is an equivalent formulation of saying that the max spectral radius has to be less than one. In general $\mu(A) \leq \rho(A)$, where $\mu(A)$ is the maximal cycle geometric mean of a non negative matrix A*

(corresponding to the cycle condition), and $\rho(A)$ is the usual spectral radius of A, cf. [Lur05].

Proposition 2.3.16 *Let $\Gamma \in \mathcal{G}^{n\times n}$ be irreducible and $\mu = \oplus$. Suppose $\Gamma_\oplus(x) \not\geq x$ for all $x \neq 0$. Then there exists a function $\sigma \in \mathcal{K}_\infty^n$ such that*

$$\sigma(t) \in \Omega(\Gamma_\oplus)$$

for all $t > 0$.

Proof. We start with an observation about ordered $\mathcal{K}_\infty$-functions: Let $\delta, \gamma \in \mathcal{K}_\infty$, so that $\gamma < \delta$. Then $\delta \circ \gamma^{-1} \in \mathcal{K}_\infty$ and $\mathrm{id} < \delta \circ \gamma^{-1}$. Also the function $\alpha := \frac{1}{2}\gamma + \frac{1}{2}\delta$ is of class $\mathcal{K}_\infty$ and satisfies $\gamma < \alpha < \delta$.

In the following we construct a diagonal operator $D = \mathrm{diag}(\eta)$, $\eta \in \mathcal{K}_\infty$, $\eta > \mathrm{id}$, such that $\Gamma_\oplus(D(s)) \not\geq s$ for $s \neq 0$. Once that we have found such an operator D, we may apply Proposition 2.3.7 to obtain σ as desired.

The matrix Γ contains a finite number of minimal subordinated cycles of length $p \geq 1$, and by Lemma 2.3.14 these are contractions, i.e., such that

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_p i_1} < \mathrm{id},$$

where

1. all $\gamma_{i_j, i_{j+1}}$, $j = 1, \ldots, p-1$, and $\gamma_{i_p i_1}$ are of class $\mathcal{K}_\infty$;
2. $i_1 > \max_{j=2,\ldots,p} i_j$; and
3. $i_j \neq i_k$ if $j \neq k$.

We are going to construct a $\mathcal{K}_\infty$ function $\eta > \mathrm{id}$ such that

$$\gamma_{i_1 i_2} \circ \eta \circ \gamma_{i_2 i_3} \circ \eta \circ \ldots \circ \gamma_{i_p i_1} \circ \eta < \mathrm{id}$$

for all such cycles. This leads to an operator $\tilde{\Gamma} = (\gamma_{ij} \circ \eta)$ satisfying $\tilde{\Gamma}_\oplus(s) \gg \Gamma_\oplus(s)$ for all $s \gg 0$ and we can write $\tilde{\Gamma}_\oplus = \Gamma_\oplus \circ D$ with $D = \mathrm{diag}(\eta)$. By construction $\Gamma_\oplus \circ D$ satisfies the cycle condition and hence by Lemma 2.3.14 we have $\Gamma_\oplus(D(s)) \not\geq s$ for all $s \neq 0$.

Now for each cycle $c = \{i_1, \ldots, i_p\}$ and every $\gamma_{ij} \neq 0$ we construct a function $\alpha_{ij}^c \in \mathcal{K}_\infty$ such that

$$\gamma_{i_1 i_2} \circ \alpha_{i_1 i_2}^c \circ \gamma_{i_2 i_3} \circ \alpha_{i_2 i_3}^c \circ \ldots \circ \gamma_{i_p i_1} \circ \alpha_{i_p i_1}^c < \mathrm{id}.$$

Then we let $\eta = \min\{\alpha_{ij}^c : c \text{ is a minimal subordinated cycle and } \gamma_{ij} \in \mathcal{K}_\infty\}$. As the minimum over a finite number of $\mathcal{K}_\infty$-functions, η is a $\mathcal{K}_\infty$-function.

Since all α_{ij}^c satisfy $\alpha_{ij}^c > \mathrm{id}$, also η satisfies $\eta > \mathrm{id}$, and for any cycle $(\gamma_{i_1 i_2}, \ldots, \gamma_{i_p i_1})$ of $\mathcal{K}_\infty$-functions in the matrix Γ, we have

$$\gamma_{i_1 i_2} \circ \eta \circ \gamma_{i_2 i_3} \circ \eta \circ \ldots \circ \gamma_{i_p i_1} \circ \eta < \mathrm{id}.$$

So it remains to construct the functions α_{ij}^c, and this is done as follows: Given a cycle $c = \{i_1, \ldots, i_p\}$ denote $z_p = \gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_p i_1}$ and $z_k = \gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_k i_{k+1}}$ for $1 \leq k < p$, where we omit the reference to the cycle c for notational simplicity. By our initial observation, we have that $\tilde{z}_p := \frac{1}{2} z_p + \frac{1}{2}\mathrm{id}$ is of class $\mathcal{K}_\infty$ and satisfies $z_p < \tilde{z}_p < \mathrm{id}$. Hence $\alpha_{i_p i_1} := z_p^{-1} \circ \tilde{z}_p$ satisfies $\alpha_{i_p i_1} > \mathrm{id}$. We find

$$\begin{aligned} & \gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_p i_1} \circ \alpha_{i_p i_1} < \mathrm{id} \\ \iff\ & z_{p-1} = \gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_{p-1} i_p} < (\gamma_{i_p i_1} \circ \alpha_{i_p i_1})^{-1} =: r_1. \end{aligned}$$

Now construct $\alpha_{i_{p-1} i_p} := z_{p-1}^{-1} \circ (\frac{1}{2} z_{p-1} + \frac{1}{2}(\gamma_{i_p i_1} \circ \alpha_{i_p i_1})^{-1}) = z_{p-1}^{-1} \circ (\frac{1}{2} z_{p-1} + \frac{1}{2} r_1)$ to obtain

$$z_{p-2} = \gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_{p-2} i_{p-1}} < (\gamma_{i_p i_1} \circ \alpha_{i_p i_1})^{-1} \circ (\gamma_{i_{p-1} i_p} \circ \alpha_{i_{p-1} i_p})^{-1} =: r_2.$$

Inductively define $\alpha_{i_{p-l} i_{p-l+1}} := z_{p-l}^{-1} \circ (\frac{1}{2} z_{p-l} + \frac{1}{2} r_l)$ and $r_l := r_{l-1} \circ (\gamma_{i_{p-l} i_{p-l+1}} \circ \alpha_{i_{p-l} i_{p-l+1}})^{-1}$ for $2 \leq l \leq p-1$ to obtain all α_{ij} in this cycle. Finally we find

$$\gamma_{i_1 i_2} \circ \alpha_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \alpha_{i_2 i_3} \circ \ldots \circ \gamma_{i_p i_1} \circ \alpha_{i_p i_1} < \mathrm{id}$$

for this cycle. □

To have the general case $\Gamma \in \mathcal{G}^{n \times n}$ with $\mu = \oplus$ it remains to consider reducible Γ.

Proposition 2.3.17 *Let $\Gamma \in \mathcal{G}^{n \times n}$ be reducible and $\mu = \oplus$. Suppose $\Gamma_\oplus(x) \not\geq x$ for all $x \neq 0$. Then there exists a function $\sigma \in \mathcal{K}_\infty^n$ such that*

$$\sigma(t) \in \Omega(\Gamma_\oplus)$$

for all $t > 0$.

Proof. The proof is very similar to that of Theorem 2.3.9, we use the same notation. The induction start is the same. Note that now we have some $\sigma_i \in \mathcal{K}_\infty^{n_i}$, $i = 1, 2$, such that

$$(\Gamma_{ii})_{\oplus_i}(\sigma_i(t)) \ll \sigma_i(t)$$

for all $t > 0$. This time we define $s : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ by

$$s(t) := \sup\{r \in \mathbb{R}_+ : (\Gamma_{12})_\oplus(\sigma_2(r)) \leq \sigma_1(t)\},$$

where we do allow $s(t) = \infty$ for some or all $t \in \mathbb{R}_+$, e.g., if $\Gamma_{12} = 0$. This function is nondecreasing and as $t \to \infty$ we have $s(t) \to \infty$. In addition s satisfies $s(0) \geq 0$ and $s(t) > 0$ for all $t > 0$. Therefore there exists a function $\tau \in \mathcal{K}_\infty$ satisfying $\tau(t) < s(t)$ for all $t > 0$. Hence for all $t > 0$ we have

$$(\Gamma_{12})_\oplus(\sigma_2(\tau(t))) \ll \sigma_1(t).$$

Now consider

$$\begin{aligned}(\pi_{I_1} \circ \Gamma_\oplus)\begin{pmatrix}\sigma_1(t)\\ \sigma_2 \circ \tau(t)\end{pmatrix} &= \Big((\Gamma_{11})_\oplus(\sigma_1(t))\Big) \oplus \Big((\Gamma_{12})_\oplus \circ \sigma_2 \circ \tau(t)\Big)\\ &\ll \sigma_1(t).\end{aligned}$$

Hence we established

$$\Gamma_\oplus \begin{pmatrix}\sigma_1(t)\\ \sigma_2 \circ \tau(t)\end{pmatrix} \ll \begin{pmatrix}\sigma_1(t)\\ \sigma_2 \circ \tau(t)\end{pmatrix} =: \sigma(t). \qquad \square$$

2.4. Dichotomy in two dimensions

Monotone maps $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, such that $T(x) \not\geq x$ for all $x \neq 0$, are a special case and give rise to less complicated dynamics than operators on higher dimensional spaces do. Here the best way to get an intuitive understanding is probably to consider Figure 2.7. Lemma 1.5.5 asserts that $T^k(x) \not\geq x$ for any $k > 0$, so that trajectories starting in $x \neq 0$ have to satisfy

$$x(k+1) = T^{k+1}(x) \notin \bigcup_{j=0}^{k} \{s \geq T^j(x)\}.$$

But still trajectories may grow unbounded and they may even oscillate between neighborhoods of the coordinate axes. We put this observation into the following theorem. Observe that the assertion of the theorem is slightly stronger: One component has to tend to zero as k tends to infinity.

Theorem 2.4.1 *Let $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ be a monotone, continuous operator and suppose there exists an operator $D : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, $D = \mathrm{diag}_2(id + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ T \not\geq id$. Suppose further, that for some $x \in \mathbb{R}_+^2$ the orbit $O(x) = \{T^k(x)\}_{k=0,1,2,\ldots}$ is unbounded. Then there exists a subsequence $\{k_j\}_{j\in\mathbb{N}_0}$ such that as $j \to \infty$ exactly one of the following holds:*

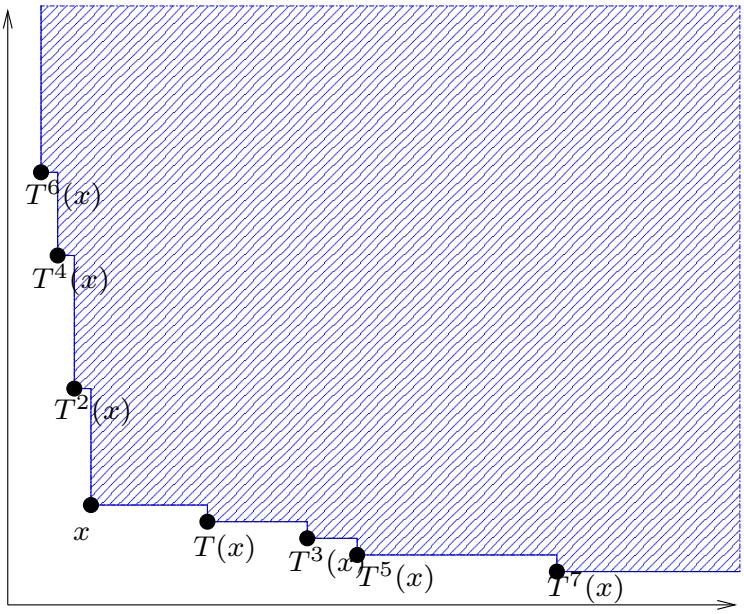


Figure 2.7.: A map $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ satisfying $T(s) \not\geq s$ for all $s \neq 0$ may not have increasing trajectories. Instead trajectories have to get closer and closer to the coordinate axes. The area where $T^{k+1}(x)$ cannot be when we already know $x, \dots, T^k(x)$ becomes bigger and bigger each step since $T^k(s) \not\geq s$ for all $s \neq 0$, $k \geq 1$, by Lemma 1.5.5.

1. $\pi_1 \circ T^{k_j}(x) \nearrow \infty$ *and* $\pi_2 \circ T^{k_j}(x) \searrow 0$,
2. $\pi_1 \circ T^{k_j}(x_0) \searrow 0$ *and* $\pi_2 \circ T^{k_j}(x_0) \nearrow \infty$.

Proof. First note that by assumption and Lemma 1.1.4 there exist operators $D_i = \operatorname{diag}_2(\mathrm{id} + \rho_i), \rho_i \in \mathcal{K}_\infty, i = 1, 2$, such that $D_2 \circ D_1 = D$ and therefore $D_1 \circ T \circ D_2 \not\geq \mathrm{id}$ by Lemma 2.2.12. By Lemma 1.5.5 we even have $(D \circ T)^L \not\geq \mathrm{id}$, which implies $D \circ T^L \not\geq \mathrm{id}$, and this in turn is equivalent to $D_1 \circ T^L \circ D_2 \not\geq \mathrm{id}$ for all $L \geq 1$ by Lemma 2.2.12.

Since $O(x)$ is unbounded, necessarily one component must be unbounded. Hence we may assume that the first component is unbounded. We may pick a sequence $\{k_j\}_{j=0}^\infty$ such that the first component of

$$(x_1(k_j), x_2(k_j))^\top := x(k_j) := T^{k_j}(x), \quad j \geq 0,$$

is strictly increasing and unbounded. So we have to show that $\pi_2 \circ T^{k_j}(x) \searrow 0$ as $j \to \infty$. The sequence $x_2(k_j)$ must be strictly decreasing, otherwise we would find a violation of the assertion of Lemma 1.5.5, as we already observed in the introduction of this subsection.

It remains to prove that $x_2(k_j)$ decreases to zero. So suppose there exists a constant $C > 0$ such that $x_2(k_j) \searrow C$. Then for $\varepsilon \in]0, \rho_1(C)[$ there exist $j' \geq 0$ and $\varepsilon' < \varepsilon$ such that

$$x(k_{j'}) = \begin{pmatrix} x_1(k_{j'}) \\ C + \varepsilon' \end{pmatrix} > 0.$$

Now for $l \geq 1$ we have

$$\begin{aligned} x(k_{j'+l}) &= T^L(x(k_{j'})) \quad \text{for some } L = L(l) \\ &= \begin{pmatrix} x_1(k_{j'+l}) \\ C + \varepsilon'' \end{pmatrix} \quad \text{for some } \varepsilon'' = \varepsilon''(l) \\ &\leq T^L(D_2(x(k_{j'}))), \end{aligned}$$

where $\varepsilon'' < \varepsilon'$. Applying D_1 to the last line, we can estimate its second component as follows

$$\begin{aligned} \pi_2 \circ D_1 \circ T^L \circ D_2(x(k_{j'})) &\geq (\mathrm{id} + \rho_1)(C + \varepsilon'') \\ &> C + \rho_1(C) > C + \varepsilon > x_2(k_{j'}) \end{aligned}$$

and for the first component we find

$$\pi_1 \circ D_1 \circ T^L \circ D_2(x(k_{j'})) > x_1(k_{j'+l}) > x_1(k_{j'}).$$

Together this gives a point $x(k_{j'}) > 0$ such that

$$D_1 \circ T^L \circ D_2(x(k_{j'})) \geq x(k_{j'})$$

in contradiction to $D_1 \circ T^L \circ D_2 \not\geq \mathrm{id}$, which has been asserted in Lemma 1.5.5. Hence we must have $C = 0$. This proves the theorem. □

With the last result in mind and reconsidering Figure 2.7 we now need a condition that guarantees that if one component of $T^k(x)$ tends to zero, then also the other component has to tend to zero. We propose a condition that is based on the notion of decay sets. This condition essentially states that ultimately, i.e., far away from the origin, the decay set Ω may not become arbitrarily thin and get arbitrarily close to one of the coordinate axes.

Given a continuous, monotone operator $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ satisfying $T(x) \not\geq x$ for all $x \neq 0$ we say that Ω is *nondegenerate* if

$$\lim_{x_2 \to \infty} \sup\{x_1 : (x_1, x_2)^\top \in \Omega\} > 0 \text{ and } \lim_{x_1 \to \infty} \sup\{x_2 : (x_1, x_2)^\top \in \Omega\} > 0,$$

where, contrary to the common definition, we define the supremum over an empty set to be zero. By Proposition 1.5.3 the set Ω is radially unbounded, so at least one of the supremums in the limits appearing above is over a nonempty set. We say a decay set Ω is *degenerate*, if it is not nondegenerate.

Theorem 2.4.2 *Let $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be a monotone, continuous operator satisfying $D \circ T(x) \not\geq x$ for some $D = \operatorname{diag}(id + \rho)$ for some $\rho \in \mathcal{K}_\infty$ and for all $x \neq 0$. Assume Ω is nondegenerate. Then the origin is globally asymptotically stable for the system*

$$x(k+1) = T(x(k)).$$

Proof. By the non degeneracy condition there is an $\varepsilon > 0$, such that, e.g.,

$$\lim_{x_1 \to \infty} \sup\{x_2 : (x_1, x_2)^\top \in \Omega\} > \varepsilon,$$

the other case being completely analogous.

By the dichotomy theorem we may assume that given $x \in \mathbb{R}^2_+$ we find some $k_0 \geq 0$ and have $\pi_2 \circ T^k(x) < \varepsilon$ for $k \geq k_0$. Now by the nondegeneracy condition this implies that there exists a point $y \in \Omega$ satisfying $T^{k_0}(x) \leq y$. Hence $O(T^{k_0}(x))$ is bounded by y, so $O(x)$ must be bounded, too. Now we established that $O(x)$ has compact closure and by Lemma 2.2.2 we find $\omega(x) = \{0\}$ implying attractivity of the origin, Lemma 2.1.3 yields stability. □

By reconsidering Example 2.1.6, we see that not every continuous, monotone operator $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ that satisfies $T(x) \not\geq x$ for all $x \neq 0$ admits a decay set Ω that is nondegenerate. Example 2.3.2 gives a matrix induced operator whose decay set also degenerate.

2.5. Monotone inequalities, solutions, and bounds on solutions

For the stability analysis of large-scale interconnected systems in the next chapter, we now develop an important tool: A result that allows to give bounds on the solutions of monotone inequalities.

From linear algebra we know that for $A \in \mathbb{R}^{n \times n}_+$ equations of the form

$$x = Ax + b, \quad x \in \mathbb{R}^n,$$

are uniquely solvable for any $b \in \mathbb{R}^n$ if $\mathrm{id} - A$ is invertible. This corresponds to $Ax \not\geq x$ for all $x > 0$, or equivalently, the spectral radius $\rho(A)$ of A must be less than 1. For a solution x of this equation we have

$$\|x\| \leq M \cdot \|b\|$$

for some constant M. Here we develop a theory that accomplishes a similar result for $\mathcal{G}^{n \times n}$-matrices, and in terms of inequalities.

Theorem 2.5.1 *Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. Assume there exists a $D = \mathrm{diag}(id+\rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma_\mu(x) \not\geq x$ for all $x \neq 0$. Then there exists a $\varphi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}_+^n$,*

$$(id - \Gamma_\mu)(w) \leq v \tag{2.12}$$

implies $\|w\| \leq \varphi(\|v\|)$.

Before we proceed with the proof of this theorem, let us note, that the converse implication is almost true:

Remark 2.5.2 *Assume $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$ and there exists a function $\varphi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}_+^n$, inequality (2.12) implies $\|w\| \leq \varphi(\|v\|)$. Then $\Gamma_\mu(x) \not\geq x$ for all $x \neq 0$:*

Indeed, if we assume there exists an $x \in \mathbb{R}_+^n$ such that $\Gamma_\mu(x) \geq x$, then we can rewrite this inequality to obtain

$$(id - \Gamma_\mu)(x) \leq 0$$

which is just (2.12) with $v = 0$. This implies $\|x\| \leq \varphi(0)$, hence it must be $x = 0$. In other words, $\Gamma_\mu(x) \not\geq x$ if $x \neq 0$.

This remark shows, that the no joint increase condition is somewhat natural, if we require $\mathcal{K}_\infty$ bounds for the solutions of nonlinear inequalities of the form (2.12).

The remark also raises the question, if the operator D is really needed, or if our result lacks a certain sharpness. Luckily, for the sharpness of our result, in general the operator D is essential, if the monotone inequality is stated as in (2.12), as the following example demonstrates.

Example 2.5.3 *We consider Γ from Example 2.2.10, that is*

$$\Gamma = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}$$

with $\gamma(t) = t(1 - e^{-t})$ of class $\mathcal{K}_\infty$ and $\mu = \Sigma$ or $\mu = \oplus$. Clearly $\Gamma_\mu(x) \not\geq x$ for any $x \neq 0$.

We show that there exists no function $\varphi \in \mathcal{K}_\infty$ such that the monotone inequality

$$(id - \Gamma_\mu)(w) \leq v \tag{2.13}$$

for $w = (w_1, w_2)^\top$, $v = (v_1, v_2)^\top$ implies $\|w\| \leq \varphi(\|v\|)$. The monotone inequality (2.13) applied to $w = (w_1, w_1)$ and $v = (v_1, v_1)$ leads to

$$w_1 e^{-w_1} = w_1 - w_1(1 - e^{-w_1}) \leq v_1 \tag{2.14}$$

Now take $v_1 = 1$, then inequality (2.14) is satisfied for any $w_1 \geq 0$. If there existed a function $\varphi \in \mathcal{K}_\infty$ such that (2.13) implied $\|w\| \leq \varphi(\|v\|)$, we would have $\|w\| \leq \varphi(\|(1,1)^\top\|)$ for any choice of $w = (w_1, w_1)^\top$. Clearly such a $\mathcal{K}_\infty$-function φ cannot exist.

Interestingly enough, we will also see that in some very general cases the operator D is not at all needed, see Theorem 2.5.4.

Proof of Theorem 2.5.1. Fix $v \in \mathbb{R}^n_+$. We first show, that for those $w \in \mathbb{R}^n_+$ satisfying (2.12) at least some components have to be bounded. To this end let

$$\begin{aligned} r^* := (D - \mathrm{id})^{-1}(v) &= \begin{pmatrix} \rho^{-1}(v_1) \\ \vdots \\ \rho^{-1}(v_n) \end{pmatrix} \\ \text{and } s^* := D(r^*) &= \begin{pmatrix} v_1 + \rho^{-1}(v_1) \\ \vdots \\ v_n + \rho^{-1}(v_n) \end{pmatrix}. \end{aligned} \tag{2.15}$$

We claim that $s \geq s^*$ implies that $w = s$ does not satisfy (2.12). So let $s \geq s^*$ be arbitrary and $r = D^{-1}(s) \geq r^*$ (as $D^{-1} \in \mathcal{K}^n_\infty$). For such s we have

$$s - D^{-1}(s) = D(r) - r \geq D(r^*) - r^* = v\,,$$

where we have used that $(D - \mathrm{id}) \in \mathcal{K}^n_\infty$. The assumption that $w = s$ satisfies (2.12) leads to

$$s \leq v + \Gamma_\mu(s) \leq s - D^{-1}(s) + \Gamma_\mu(s)\,,$$

or equivalently, $0 \leq \Gamma_\mu(s) - D^{-1}(s)$. This implies for $r = D^{-1}(s)$ that

$$r \leq \Gamma_\mu \circ D(r)\,,$$

in contradiction to $\Gamma_\mu \circ D(x) \not\geqq x$ for all $x \neq 0$. This shows that the set of $w \in \mathbb{R}^n_+$ satisfying (2.12) does not intersect the set

$$Z_1 := \{w \in \mathbb{R}^n_+ \mid w \geq s^*\}\,.$$

Assume now that $w \in \mathbb{R}^n_+$ satisfies (2.12). Let $s^1 := s^*$. If $s^1 \not\geqq w$, then there exists an index set $I_1 \subset \{1, \ldots, n\}$, possibly depending on w, such that

$$w_i > s^1_i, \text{ for } i \in I_1 \quad \text{and} \quad w_i \leq s^1_i, \qquad \text{for } i \in I_1^c := \{1, \ldots, n\} \setminus I_1\,.$$

So from (2.12) and using subadditivity of monotone aggregation functions

$$\pi_{I_1} \circ \Gamma_\mu(w) \leq (\Gamma_{I_1 I_1})_\mu(w|_{I_1}) + (\Gamma_{I_1 I_1^c})_\mu(w|_{I_1^c})$$

we obtain

$$w_{I_1} - ((\Gamma_{I_1 I_1})_\mu(w_{I_1}) + (\Gamma_{I_1 I_1^c})_\mu(w_{I_1^c})) \le v_{I_1}.$$

Hence we have in particular

$$\begin{aligned} w_{I_1} - (\Gamma_{I_1 I_1})_\mu(w_{I_1}) &\le v_{I_1} + (\Gamma_{I_1 I_1^c})_\mu(s^1_{I_1^c}) \\ &\le \underbrace{D_{I_1} \circ (D_{I_1} - \mathrm{id}_{I_1})^{-1}}_{>\mathrm{id}} \circ (v_{I_1} + (\Gamma_{I_1 I_1^c})_\mu(s^1_{I_1^c})) =: s^2_{I_1}. \end{aligned} \tag{2.16}$$

Note that $(\Gamma_{I_1 I_1})_\mu$ satisfies $(\Gamma_{I_1 I_1})_\mu \circ D_{I_1}(x) \ngeq x$ for all $x \neq 0$. Thus, arguing just as before we obtain that $w_{I_1} \ge s^2_{I_1}$ is not possible. Hence some more components of w must be bounded.

We proceed inductively, defining

$$I_{j+1} \subsetneqq I_j, \quad I_{j+1} := \{i \in I_j : w_i > s_i^{j+1}\},$$

with $I^c_{j+1} := \{1, \dots, n\} \setminus I_{j+1}$ and

$$s^{j+1}_{I_j} := D_{I_j} \circ (D_{I_j} - \mathrm{id}_{I_j})^{-1} \circ (v_{I_j} + (\Gamma_{I_j I_j^c})_\mu(s^j_{I_j^c})).$$

This nesting will end after at most $n-1$ steps: There exists a maximal $k \le n$, such that

$$\{1, \dots, n\} \supsetneqq I_1 \supsetneqq \dots \supsetneqq I_k \neq \emptyset$$

and all components of w_{I_k} are bounded by the corresponding components of $s^{k+1}_{I_k}$. Let

$$s_\zeta := \max\{s^*, \iota_{I_1}(s^1), \dots, \iota_{I_k}(s^k)\},$$

with antiprojections ι_{I_l}, $l = 1, \dots, k$. If we denote by $[M]^n$ the n-fold composition $M \circ \dots \circ M$, then for w satisfying (2.12) we clearly have

$$w \le s_\zeta \le [D \circ (D - \mathrm{id})^{-1} \circ (\mathrm{id} + \Gamma_\mu)]^n(v)$$

and the term on the very right hand side does not depend on any particular choice of nesting of the index sets. Hence every w satisfying (2.12) also satisfies

$$w \le [D \circ (D - \mathrm{id})^{-1} \circ (\mathrm{id} + \Gamma_\mu)]^n \left(\|v\|_{\max}, \quad \dots, \quad \|v\|_{\max}\right)^\top$$

and taking the maximum-norm on both sides yields

$$\|w\|_{\max} \le \tilde{\varphi}(\|v\|_{\max})$$

for some function $\tilde{\varphi}$ of class $\mathcal{K}_\infty$ and the same inequality for any other norm on $\mathbb{R}^n_+$ for some $\mathcal{K}_\infty$-function φ. This completes the proof of the theorem. □

The solvability theorem can be formulated without the D, too, but then we have to restrict the choices of μ to $\oplus$. Interestingly, in this case solvability is equivalent to the no joint increase condition.

Theorem 2.5.4 *Let $\Gamma \in \mathcal{G}^{n\times n}$. Then the following are equivalent:*

1. *$\Gamma_\oplus(x) \not\geq x$ for all $x \neq 0$.*
2. *There exists a function $\varphi \in \mathcal{K}_\infty$ such that for any $b \in \mathbb{R}^n_+$ the solutions x of the inequality*
$$x \leq \Gamma_\oplus(x) \oplus b \tag{2.17}$$
satisfy $\|x\| \leq \varphi(\|b\|)$.

At first sight it may seem that this result contradicts Example 2.5.3. The important difference here is the formulation of the monotone inequality, which is not equivalent to
$$(x - \Gamma_\oplus(x)) \leq b. \tag{2.18}$$
In fact, by Example 2.5.3 we see that (2.18) is strictly weaker than (2.17).

Proof. First we assume that assertion 1. holds and derive assertion 2. Consider

$$\begin{aligned} x &\leq \Gamma_\oplus(x) \oplus b \\ &\leq \Gamma_\oplus(\Gamma_\oplus(x) \oplus b) \oplus b \\ &= \Gamma_\oplus(\Gamma_\oplus(x)) \oplus \Gamma_\oplus(b) \oplus b \\ &\leq \ldots \leq \Gamma^k_\oplus(x) \oplus \bigoplus_{l=0}^{k-1} \Gamma^l_\oplus(b) \end{aligned} \tag{2.19}$$

for all $k \geq 1$.

By Theorem 2.2.9 we know that $O(s) = \{\Gamma^k_\oplus(s)\}_{k\geq 0}$ is bounded for every $s \in \mathbb{R}^n_+$ and $\Gamma^k_\oplus(s) \to 0$ as $k \to \infty$.

Hence the term $\Gamma^k_\oplus(x)$ tends to zero as $k \to \infty$ and for fixed b the term $\bigoplus_{l=0}^{k-1} \Gamma^l_\oplus(b)$ is bounded in k.

We obtain from (2.19) for $k \to \infty$ that $x \leq \bigoplus_{l=0}^{\infty} \Gamma^l_\oplus(b)$, so that $\|x\|$ is clearly bounded by some $\mathcal{K}_\infty$-function φ of $\|b\|$: To this end note that the supremum $\bigoplus_{l=0}^{\infty} \Gamma^l_\oplus(b)$ is actually attained in finitely many steps, i.e., for some $k_0 \geq 0$ such that $\bigoplus_{l=0}^{\infty} \Gamma^l_\oplus(b) = \bigoplus_{l=0}^{k_0} \Gamma^l_\oplus(b)$. This is a consequence of the fact that $\{\Gamma^l_\oplus(b)\}_{l\geq 0}$ is a null sequence, and the supremum over a null sequence in $\mathbb{R}^n_+$ with respect to the standard partial order is attained after finitely many steps.

Now we prove the converse implication, so let us assume assertion 2. is true. Assume that there exists an $x \in \mathbb{R}^n_+$ such that $\Gamma_\oplus(x) \geq x$. Then by 2. for $b = 0$ we have $x \leq \Gamma_\oplus(x) \oplus 0$ and hence $\|x\| \leq \varphi(0)$. This implies $x = 0$. So if $y \neq 0$ then $\Gamma_\oplus(y) \not\geq y$. □

As we already have seen, the formulation of Theorems 2.5.1 and 2.5.4 are very similar. The implication in (2.12) could also be stated as

$$w \leq \Gamma_\mu(w) + v \implies \|w\| \leq \varphi(\|v\|).$$

Observe that in this formulation there is a '+', whereas in the corresponding formulation in (2.17) there is a '$\oplus$' in the same position. We know that these are special case of monotone aggregation functions and it is natural to ask, if there is a 'common' version incorporating a general monotone aggregation function μ.

Unfortunately we can only partly answer this positively, that is to say, at the expense of an additional assumption that guarantees the convergence of an infinite sum that arises by repeated application of the weak triangular inequality.

For $\mu \in \mathrm{MAF}^n_{n+1}$ and $\Gamma = (\gamma_{ij})^n_{i,j=1} \in \mathcal{G}^{n\times n}$ define

$$\Gamma_\mu(s) := \mu(\Gamma(s), 0),$$

where 0 denotes the $n \times 1$-zero column and $\Gamma(s)$ is $(\gamma_{ij}(s_j))^n_{i,j=1}$ for $s \in \mathbb{R}^n_+$. For vectors $b \in \mathbb{R}^n_+$ we define $b_\mu := \mu(b) := \mu(0, b)$. Using this new notation we find the following generalization of the preceding theorem, where we assume that $b_\mu = b$ for all $b \in \mathbb{R}^n_+$.

Proposition 2.5.5 *Let $\Gamma \in \mathcal{G}^{n\times n}$ have no zero rows and $\mu \in \mathrm{MAF}^n_{n+1}$ and assume that $b_\mu = b$ for all $b \in \mathbb{R}^n_+$. Assume there exists a $D = \mathrm{diag}(id + \rho)$, $\rho \in \mathcal{K}_\infty$, such that $D \circ \Gamma_\mu(x) \not\geq x$ for all $x \neq 0$.*

Suppose for some $w \in \Omega(\Gamma_\mu \circ D)$ that

$$\limsup_{k\to\infty} \frac{\pi_i \circ (\Gamma_\mu \circ D)^k(w)}{\pi_i \circ (\Gamma_\mu \circ D)^{k-1}(w)} < 1 \tag{2.20}$$

for $i = 1, \ldots, n$. Then there exists a function $\varphi \in \mathcal{K}_\infty$ such that for any $b \in \mathbb{R}^n_+$ the solutions x of the inequality

$$x \leq \mu(\Gamma(x), b)$$

satisfy $\|x\| \leq \varphi(\|b\|)$.

In the statement of the proposition, since Γ has no zero rows, by Lemma 1.3.2 the operator Γ_μ is strictly increasing. Hence $(\Gamma_\mu \circ D)^k(\Omega(\Gamma_\mu \circ D)) \subset \Omega(\Gamma_\mu \circ D)$ by Lemma 1.5.8 for $k \geq 0$. In particular $w \in \Omega(\Gamma_\mu \circ D)$ implies not only $w \gg 0$ but also $(\Gamma_\mu \circ D)^k(w) \gg 0$, so that the denominator in (2.20) will never become zero.

Proof. First we factorize $D = D_1 \circ D_2$, where $D_i = \operatorname{diag}(\mathrm{id} + \rho_i)$, $\rho_i \in \mathcal{K}_\infty$, $i = 1, 2$, by Lemma 1.1.4. Then since $D(x) \geq D_1(x)$ it follows that $\Gamma_\mu \circ D_1(x) \not\geq x$ for all $x \neq 0$. Define $\tilde{D}_1 := \operatorname{diag}(\mathrm{id} + \rho_1^{-1})$. Then we have

$$(\Gamma_\mu \circ D_1)^k \circ \Gamma_\mu(x + y) \leq (\Gamma_\mu \circ D_1)^k \circ \Gamma_\mu \circ D_1(x) + (\Gamma_\mu \circ D_1)^k \circ \Gamma_\mu \circ \tilde{D}_1(y)$$

for all $x, y \in \mathbb{R}^n_+$ and $k \geq 0$ by the weak triangular inequality and subadditivity of MAFs.

Consider

$$\begin{aligned} x &\leq \mu(\Gamma(x), b) \\ &\leq \Gamma_\mu(x) + b \\ &\leq \Gamma_\mu(\Gamma_\mu(x) + b) + b \\ &\leq \Gamma_\mu \circ D_1 \circ \Gamma_\mu(x) + \Gamma_\mu \circ \tilde{D}_1 \circ (b) + b \\ &\leq \ldots \leq (\Gamma_\mu \circ D_1)^k \circ \Gamma_\mu(x) + (\Gamma_\mu \circ D_1)^{k-1} \circ \Gamma_\mu \circ \tilde{D}_1 \circ (b) + \\ &\qquad \ldots + \Gamma_\mu \circ D_1 \circ \Gamma_\mu \circ \tilde{D}_1 \circ (b) + \Gamma_\mu \circ \tilde{D}_1 \circ (b) + b \end{aligned} \tag{2.21}$$

for all $k \geq 1$.

We know that $O(s) = \{(\Gamma_\mu \circ D_1)^k(s)\}_{k \geq 0}$ is bounded for every $s \in \mathbb{R}^n_+$ and hence $(\Gamma_\mu \circ D_1)^k(x) \to 0$ as $k \to \infty$ by Theorem 2.2.9.

Next we show that the term $\sum_{l=0}^{k-1} (\Gamma_\mu \circ D_1)^l \circ \Gamma_\mu \circ \tilde{D}_1(b)$ is bounded as $k \to \infty$.

By Theorem 2.3.9 the set Ω is jointly unbounded, hence for any given b there exists an $s \in \Omega$ such that $(\Gamma_\mu \circ \tilde{D}_1)(b) \leq s$.

Now for any $l \geq 0$ we have $(\Gamma_\mu \circ D_1)^l \circ \Gamma_\mu \circ \tilde{D}_1(b) \leq (\Gamma_\mu \circ D_1)^l(s)$. So by Theorem 2.2.9 and employing $D_2 \circ (\Gamma_\mu \circ D_1)(x) \not\geq x$ for all $x \neq 0$, as $l \to \infty$ we have $(\Gamma_\mu \circ D_1)^l(s) \to 0$. In particular, there exists an $L \geq 1$ such that for all $l \geq L$ we have $(\Gamma_\mu \circ D_1)^l(s) \leq w$, where w is given by our assumptions and satisfies $w \gg 0$, since, using the definition of $w \in \Omega(\Gamma_\mu \circ D)$ we have $0 \leq \Gamma_\mu \circ D(w) \ll w$. Clearly

$$\sum_{l=0}^{k} (\Gamma_\mu \circ D_1)^l \circ \Gamma_\mu \circ \tilde{D}_1(b) \leq \sum_{l=0}^{k} (\Gamma_\mu \circ D_1)^l(s), \quad k \geq 1,$$

and

$$\sum_{l=0}^{k}(\Gamma_\mu \circ D_1)^l(s) \leq \sum_{l=0}^{L-1}(\Gamma_\mu \circ D_1)^l(s) + \sum_{l=0}^{k-L}(\Gamma_\mu \circ D_1)^l(w), \quad k \geq L.$$

Now as $k \to \infty$ the summands in the right most sum in the last inequality are of the form $(\Gamma_\mu \circ D_1)^l(w)$. Considering each component individually, by (2.20) we may apply the quotient criterion for infinite sums: The sum $\sum_{l=0}^{\infty} \pi_i \circ (\Gamma_\mu \circ D_1)^l(w)$ converges if there exists a $0 < q < 1$ such that for some $K > 0$ for all $l > K$,

$$\frac{\pi_i \circ (\Gamma_\mu \circ D_1)^{l+1}(w)}{\pi_i \circ (\Gamma_\mu \circ D_1)^l(w)} < q,$$

which is assured by (2.20).

We obtain from (2.21) for $k \to \infty$ that $x \leq \sum_{l=0}^{\infty}(\Gamma_\mu \circ D_1)^l \circ \Gamma_\mu \circ \tilde{D}_1(b) =: f(b)$. The function f is clearly nondecreasing in b and satisfies $f(0) = 0$.

We have to show that f is continuous at the origin. Then $t \mapsto \|f(t \cdot e)\|$, with $e = (1, \ldots, 1)^\top$, can be bounded from above by a $\mathcal{K}_\infty$ function, so that $\|x\|$ is clearly bounded by some $\mathcal{K}_\infty$-function φ of $\|b\|$:

Basically, we have to assure that, as b tends to zero, also $f(b)$ has to tends to zero. We do so by a summand-dropping argument.

To this end note, firstly, that if $\{a_k\} \subset \mathbb{R}^n_+$ is a null sequence such that $\sum_{k=0}^{\infty} a_k < \infty$, then $\lim_{l\to\infty} \sum_{k=l}^{\infty} a_k = 0$.

Secondly, for any $l > 0$, a point w' defined by $w' = (\Gamma_\mu \circ D)^l(w)$ satisfies $w' \in \Omega(\Gamma_\mu \circ D)$ and (2.20). In particular $w' \gg 0$.

Consequently, for any decreasing null sequence $\{b_k\} \subset \mathbb{R}^n_+$ satisfying $\Gamma_\mu \circ \tilde{D}_1(b_k) \ll w$ for all $k \geq 0$ there exists a decreasing null sequence w'_k of points in $\Omega(\Gamma_\mu \circ D)$ satisfying (2.20), so that $\Gamma_\mu \circ \tilde{D}_1(b_k) \ll w'_k$ for all $k \geq 0$.

So we can bound

$$f(b_k) \leq \sum_{l=0}^{\infty}(\Gamma_\mu \circ D_1)^l(w'_k) \leq \sum_{l=0}^{\infty}(\Gamma_\mu \circ D)^l(w'_k).$$

The right hand side of this estimate is now of a particular form: As $k \to \infty$, the majorizing summands are 'dropped', as for smaller b_k we may also employ smaller w'_k, which can be chosen to be of the form $(\Gamma_\mu \circ D)^k(w)$. This essentially is the same as dropping the first summands on the right hand side of the above estimate. Therefore, the right hand side of our estimate on $f(b_k)$ tends to zero as $k \to \infty$. This proves that f is continuous at the origin. □

2.6. A local version of Theorem 2.5.1

In some cases the condition that there exists an operator $D = \mathrm{diag}(\mathrm{id} + \rho)$ for some $\rho \in \mathcal{K}_\infty$ such that $(D \circ \Gamma_\mu)(s) \not\geq s$ for **all** $s \neq 0$ is too restrictive, or for numerical reasons, is only assured locally, i.e., in a neighborhood of the origin.

In this section we show that if $\Gamma(s) \not\geq s$ for all $s \neq 0$, $s \leq w$ for some $w \gg 0$, then at least a local version of Theorem 2.5.1 holds. This will be eminently useful for the numerical implementation of our results in Chapter 4.

Again, we consider the system

$$x(k+1) = T(x(k)). \tag{2.22}$$

Lemma 2.6.1 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone. Assume there exists a $w \geq 0$ such that for the ω-limit set of w, corresponding to the system (2.22), we have $\omega(w) = \{0\}$. Then $T(s) \not\geq s$ for all $s \in [0, w]$, $s \neq 0$.*

Proof. We prove this indirectly. Suppose there exists an $r \neq 0, r \in [0, w]$, such that $T(r) \geq r$. By monotonicity of T it follows that $T^k(r) \geq \ldots \geq T(r) \geq r$ for all $k \geq 1$. On the other hand, the inequality $r \leq w$ implies $0 \leq T^k(r) \leq T^k(w)$ for all $k \geq 0$. Since $T^k(w) \to 0$ we also have $T^k(r) \to 0$ as $k \to \infty$, a contradiction. Therefore such an r can't exist, proving $T(r) \not\geq r$ for all $r \in [0, w]$, $r \neq 0$. □

As a consequence, we obtain an even stronger implication, if we pick w in the previous lemma such that we have $T(w) \ll w$ in the first place. If T is monotone, continuous, and such that $T(s) \not\geq s$ for all $s \in \mathbb{R}^n_+, s \neq 0$, then such a w always exists with $|w|$ arbitrary large by Proposition 1.5.3.

Theorem 2.6.2 *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and strictly increasing. Assume there exists a $w \geq 0$ such that $T(w) \ll w$ and for the ω-limit set of w, corresponding to the system (2.22), we have $\omega(w) = \{0\}$. Then for any $w^1 \in]T(w), w[$ there exists a $\rho \in \mathcal{K}_\infty$, such that for $D = \mathrm{diag}(id + \rho)$ we have $D \circ T(s) \not\geq s$ for all $s \in [0, w^1]$, $s \neq 0$.*

More informally stated, $\Gamma_\mu(s) \not\geq s$ locally for $s \neq 0$ implies the existence of diagonal operator D such that $(D \circ \Gamma_\mu)(s) \not\geq s$ locally for $s \neq 0$.

Proof. Pick $w^1 \in]T(w), w[$. For future reference denote $w^0 = w$. By monotonicity of T we know $T^k(w^1) \leq T^k(w)$ for all $k \geq 0$. Hence $T^k(w^1) \to 0$ as $k \to \infty$. We construct D in two steps. First we define D on a discrete set, then we interpolate from this discrete step to an operator on $[0, w^1]$, this operator is the restriction of an operator of the form $\mathrm{diag}(\mathrm{id} + \rho)$ on $\mathbb{R}^n_+$ to the interval $[0, w^1]$.

To this end extend w^0, w^1 to a sequence $\{w^k\}_{k\geq 0}$ by

$$w^{k+1} := T(w^{k-1}), \quad k \geq 1.$$

Since $T(w^0) \ll w^1 \ll w^0$ and T is strictly increasing we find $T^{k+1}(w^0) \ll T^k(w^1) \ll T^k(w^0)$, hence $w^{k+1} \ll w^k$ for all $k \leq 0$, i.e., the sequence $\{w^k\}_{k\geq 0}$ is strictly decreasing.

Since $\{w^k\}_{k\geq 0} \subset O(w^1) \cup O(w^0)$ the accumulation points of $\{w^k\}_{k\geq 0}$ are a subset of union of the sets of accumulation points of $O(w^1)$ and $O(w^0)$, respectively. But these are $\omega(w^1) = \omega(w^0) = \{0\}$. Hence $\{w^k\}_{k\geq 0}$ is a null sequence.

Now define $\tilde{D}$ inductively by $\tilde{D}(w^{k+1}) := w^k$ for $k \geq 1$, i.e., $\tilde{D}(T(w^k)) = w^{k+1}$ for $k \geq 0$. Since $\{w^k\}_{k\geq 0}$ is strictly decreasing, the minimum of the differences of the components of $\tilde{D}(w^{k-1})$ and w^{k-1} are positive and hence $\tilde{\alpha}^{k-1} := \min_i(\tilde{D}(w^{k-1}) - w^{k-1})_i > 0$ for all $k \geq 1$.

We can bound the sequence $\{\tilde{\alpha}^k\}_{k\geq 0}$ from below by a strictly decreasing null sequence $\{\alpha^k\}_{k\geq 0}$. Define the sequence $\{r^k\}_{k\geq 0}$ by $r^k := \sum_i w_i^k$. Then this sequence is a strictly decreasing null sequence, too.

Now construct $\tilde{\rho} : [0, r^0] \to \mathbb{R}_+$ by linear interpolation

$$\tilde{\rho}(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{(\alpha^k - \alpha^{k+1})}{(r^k - r^{k+1})}(t - r^{k+1}) + \alpha^{k+1} & \text{if } t \in]r^{k+1}, r^k] \text{ for a } k \geq 0. \end{cases}$$

By the strict monotonicity of the sequences $\{\alpha^k\}_{k\geq 0}$ and $\{r^k\}_{k\geq 0}$ the function $\tilde{\rho}$ is strictly increasing. By construction it is also continuous. Therefore it can be extended to a function ρ of class $\mathcal{K}_\infty$.

Remember that the operator $\tilde{D}$ satisfies $w^{k+2} \ll \tilde{D}(w^{k+2}) = w^{k+1} \ll w^k$ for all $k \geq 0$. We now define a diagonal operator $D = \mathrm{diag}(\mathrm{id} + \rho)$. Denote by e the vector of ones. The operator D satisfies the inequality

$$w^k \ll D(w^k) \leq w^k + \rho(r^k)e = w^k + \alpha^k e \leq w^k + (\tilde{D}(w^k) - w^k) = \tilde{D}(w^k)$$

for all $k \geq 0$. It follows that

$$0 \leq (D \circ T)(w^k) \leq (\tilde{D} \circ T)(w^k) = \tilde{D}(w^{k+2}) = w^{k+1} \ll w^k \tag{2.23}$$

for all $k \geq 0$. In particular we have

$$(D \circ T)(w^1) \leq (\tilde{D} \circ T)(w^1) = \tilde{D}(w^3) = w^2. \tag{2.24}$$

The operator $D \circ T$ is monotone and hence, by repeated application of $(D \circ T)$ to (2.24), and utilizing (2.23), we obtain

$$(D \circ T)^k(w^1) \leq (D \circ T)^{k-1}(w^2) \leq \ldots \leq (D \circ T)(w^k) \leq w^{k+1}$$

for all $k \geq 0$. Hence $(D \circ T)^k(w^1) \to 0$ as $k \to \infty$. By Lemma 2.6.1 we obtain that $(D \circ T)(s) \not\geq s$ for all $s \neq 0, s \in [0, w^1]$. □

Using the last theorem we can prove the following local version of the solvability theorem.

Theorem 2.6.3 *Let $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$. Assume there exists a $w' \geq 0$ such that $\Gamma_\mu(w') \ll w'$ and for the ω-limit set of w' with respect to (2.5) we have $\omega(w') = \{0\}$. Then there exists a $w^* \in]0, w'[$ such that there exists a $\varphi \in \mathcal{K}_\infty$ so that for all $w, v \in \mathbb{R}_+^n$, with $w \leq w^*$,*

$$(id - \Gamma_\mu)(w) \leq v \tag{2.25}$$

implies $\|w\| \leq \varphi(\|v\|)$.

Proof. The proof combines the results of Theorem 2.6.2 and Theorem 2.5.1.

As asserted by Theorem 2.6.2, for all $\hat{w} \in]\Gamma_\mu(w'), w'[$ there exists a $\rho_{\hat{w}} \in \mathcal{K}_\infty$, such that for $D_{\hat{w}} = \mathrm{diag}(\mathrm{id} + \rho_{\hat{w}})$ we have $D_{\hat{w}} \circ \Gamma_\mu(s) \not\geq s$ for all $s \neq 0$, $s \in [0, \hat{w}]$. This is equivalent as to say that $\Gamma_\mu \circ D_{\hat{w}}(s) \not\geq s$ for all $s \neq 0$, $s \in [0, D_{\hat{w}}^{-1}(\hat{w})]$, as the calculation in the proof of Lemma 2.2.12 yields. Now pick such a $\hat{w}$ and define $w^* := D_{\hat{w}}^{-1}(\hat{w})$.

The remainder of the proof goes along the lines of the proof of Theorem 2.5.1, just note that every occurrence of $\Gamma_\mu \circ D(x) \not\geq x$ for all $x \neq 0$ in that proof may be replaced by the local counterpart $(\Gamma_\mu \circ D)(s) \not\geq s$ for all $s \in [0, w^*]$, $s \neq 0$, because in the original proof only local statements are used. □

2.7. Notes and references

Monotone dynamical systems have been studied in [HS05, AS04a, AS04b, Smi95], but also in [LaS76, KH95]. Discrete time monotone systems have been of interest in [HS05, Smi95]. The PhD dissertation [Pot02] and the conference article [PAQ03] based on it consider idempotent analysis of certain monotone operators and also include graph theoretical considerations. The authors of these two publications find a solvability result for $\mu = \oplus$ very similar to Theorem 2.5.4, directly with an application to networks of ISS systems in mind. They also state that a sufficient condition for their result is that all cycles are contractions.

It is interesting to note that for $n \geq 3$ we were not able to prove something similar to the dichotomy in Theorem 2.4.1, i.e., that always one component

of any trajectory for $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ has to tend to zero. Nevertheless, it should be possible to extend the notion of nondegeneracy to $n \geq 3$ and prove something similar to Theorem 2.4.2, i.e., with the requirement that one component of each trajectory has to tend to zero.

In [Tee05] the result Theorem 2.5.4 is proved for the cycle condition and in a different way. The argument is based on the contraction properties of the cycles and in that sense more direct, for our proof is based on results developed in the preceding chapter. The result in [Tee05] is also applied to the stability analysis of ISS networks.

3. Input-to-output stability for interconnected systems

Stability of equilibrium states of dynamical systems is a important concept in control theory. Stability means that small perturbations of the initial state effectively do not lead to a quantitatively significant change of behavior of the system, i.e., for initial states near that equilibrium state the systems remains near the equilibrium. In most applications of control theory this is a desirable behavior of a system. But this behavior is not always *a priory* given.

As there exists a variety of stability concepts in the control and dynamical systems literature, so there exist several ways to investigate stability properties of dynamical systems. Here we are interested in stability of large scale systems.

For large scale linear systems the well developed and mature theory of linear dynamical systems applies. For the nonlinear case there exist several different approaches. As it turns out, the stability concepts from the linear theory do in general not possess canonical counterparts in the nonlinear theory. Therefore the developments in the nonlinear theory spread in many different directions.

A relatively young stability concept for dynamical systems with inputs has been introduced in 1989 and is termed input-to-state stability (ISS), see [Son89]. This concept carries some of the nice properties of stability concepts of linear systems to nonlinear systems [Son01, GSW99]. An extension of this concept to systems with outputs is input-to-output stability (IOS), see [JTP94]. These two stability concepts share the following two important structural properties: Cascades of ISS systems are again ISS and, provided that a small gain condition is satisfied, feedback loops of ISS systems are again ISS. Similar results are true for IOS systems, but for feedback loops in addition another property is required: The unboundedness observability property. This property ensures that the state of a system remains bounded if its output and input remain bounded.

Using the cascade and the small gain rule, it is possible to determine that large scale interconnections (networks) of IOS/ISS systems are also IOS/ISS. By a stepwise simplification of the network by iterative application of either

the cascade or the small gain rule, we may end up with a single system, that is IOS/ISS.

Of course this procedure yields only a sufficient stability test: The step-by-step simplification algorithm may fail to prove stability, even if the interconnection of interest is IOS/ISS. This is a drawback that all stability tests of large scale systems based on worst case estimates share, since effects of negative feedback can not be taken account of. But even worse, for this iterative reduction procedure there is no canonical place to start with the simplification. And it is not clear, whether all possible choices of steps to simplify a network lead to the same result.

Example 3.0.1 *We are going to see that a step by step simplification of a network of ISS systems is cumbersome and due to applications of the weak triangular inequality, the estimates lose sharpness at every step.*

Let us consider a simple network consisting of three interconnected systems

$$\dot{x}_i = f_i(x_1, x_2, x_3), \quad i = 1, 2, 3.$$

Here we assume $x_i \in \mathbb{R}^{N_i}$, *and* f_i *to be locally Lipschitz continuous for* $i = 1, \ldots, 3$. *Assume further that for each system there holds an estimate of the form*

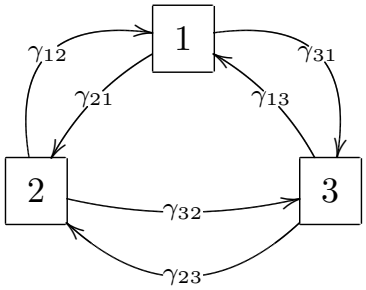
$$\|x_i(t)\| \leq \beta_i(\|x_i^0\|, t) + \sum_{j \neq i} \gamma_{ij}(\|x_{j,[0,t]}\|), \tag{3.1}$$

where the gains γ_{ij}, $i, j = 1, 2, 3$, *are in class* $\mathcal{G}$ *and* β_i, $i = 1, 2, 3$, *are class* $\mathcal{KL}$ *functions. This is a simple case of an input-to-state stability (ISS) estimate.*

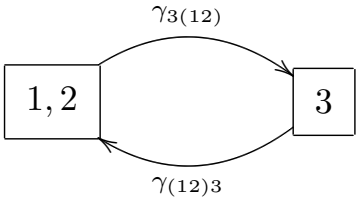
The known small gain theorem from [JTP94] *states, that if two ISS systems are feedback interconnected with gains* $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ *then the feedback interconnection is ISS if there exist* $\rho, \kappa \in \mathcal{K}_\infty$ *such that the gains satisfy the small gain condition*

$$\gamma_1 \circ (id + \rho) \circ \gamma_2 \circ (id + \kappa) < id. \tag{3.2}$$

We try to apply this condition repeatedly to check whether the network of ISS systems above with the following topology is also ISS:



Assume the gains γ_{12} and γ_{21} satisfy the small gain condition (3.2)*. Then we can simplify the network by aggregating system* 1 *and system* 2 *into one new system* $(1,2)$*:*



The calculation of the resulting gains depends on the norm that we choose for the compound state $(x_1^\top, x_2^\top)^\top$*. Let us assume that we want to use the Euclidean norm* $\|(x_1^\top, x_2^\top)^\top\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$*. For simplicity we forget about the transient bounds* β_i*,* $i = 1,2,3$*, i.e., we consider the gains as* t *tends to* ∞*. With a slight abuse of notation by* x_i *we denote* $\limsup_{t\to\infty} \|x_i(t)\|$ *for* $i = 1,2,3$*. Then by the estimates* (3.1) *and the small gain condition we obtain*

$$\begin{aligned} x_1 &\le \gamma_{12}(x_2) + \gamma_{13}(x_3) \\ &\le \gamma_{12}\big(\gamma_{21}(x_1) + \gamma_{23}(x_3)\big) + \gamma_{13}(x_3) \end{aligned}$$

by the small gain condition (3.2) there exist $\rho_1, \kappa_1 \in \mathcal{K}_\infty$ *such that, by the weak triangle inequality, this is*

$$\le \underbrace{\big(\gamma_{12} \circ (id + \rho_1)\big)\big(\gamma_{21}(x_1)\big)}_{(*)} + \big(\gamma_{12} \circ (id + \rho_1^{-1})\big)\big(\gamma_{23}(x_3)\big) + \gamma_{13}(x_3)$$

and the part denoted by $(*)$ *is less or equal to* $(id+\kappa_1)^{-1}(x_1) = (id-\tilde{\kappa}_1)(x_1)$ *for some* $\tilde{\kappa}_1 \in \mathcal{K}_\infty$ *by Lemma 1.1.5, yielding*

$$x_1 \le \tilde{\kappa}_1^{-1}\Big(\big(\gamma_{12} \circ (id + \rho_1^{-1})\big)\big(\gamma_{23}(x_3)\big) + \gamma_{13}(x_3)\Big).$$

Similarly for x_2 *we obtain*

$$x_2 \le \tilde{\kappa}_2^{-1}\Big(\big(\gamma_{21} \circ (id + \rho_2^{-1})\big)\big(\gamma_{13}(x_3)\big) + \gamma_{23}(x_3)\Big).$$

The gain from x_3 *to* $(x_1^\top, x_2^\top)^\top$ *with respect to the Euclidean norm can then be estimated as*

$$\gamma_{(12)3}(x_3) = \sqrt{\Big(\tilde{\kappa}_1^{-1}\Big(\big(\gamma_{12} \circ (id + \rho_1^{-1})\big)\big(\gamma_{23}(x_3)\big) + \gamma_{13}(x_3)\Big)\Big)^2 + \Big(\tilde{\kappa}_2^{-1}\Big(\big(\gamma_{21} \circ (id + \rho_2^{-1})\big)\big(\gamma_{13}(x_3)\big) + \gamma_{23}(x_3)\Big)\Big)^2}.$$

The gain from the compound system $(1,2)$ *to system* 3 *is simpler to calculate, here we have*

$$x_3 \leq \gamma_{31}(x_1) + \gamma_{32}(x_2) \leq \gamma_{31}(\|(x_1^\top, x_2^\top)^\top\|) + \gamma_{32}(\|(x_1^\top, x_2^\top)^\top\|)$$

so that we may choose

$$\gamma_{3(12)}(t) = \gamma_{31}(t) + \gamma_{32}(t).$$

Again we face a feedback situation, and if the network would be larger, we can imagine that the terms for the gains between compounds become more and more complicated after each simplification step. Already for three interconnected systems we see this effect, how does this scale for a large number of systems? Also, each time there is ambiguity in the choice of the functions ρ *and* κ*, and it is not always obvious whether or not it is a good idea to choose them as 'small' as possible. Depending on the choices we make, the successive simplification of the network might be successful or not.*

If the gains in the example are all linear functions $\gamma_{ij}(t) = g_{ij}t$, then they give rise to a nonnegative matrix $G = (g_{ij})$, see also Section 3.3.2. Using μ-analysis as proposed in [KHP06] it is possible to show that if the systems themselves are linear, and $\rho(G) < 1$ then the origin is globally asymptotically stable for the interconnected system. By a result in [Lur05] it is known, that even if all cycles in the gain matrix G are less than one, then still the spectral radius of G might be greater than one.

So already in the linear case the idea of extending the small gain condition to all cycles, i.e., such that for every $g_{i_1 i_2}, \ldots, g_{i_{k-1} ik}$ with $i_k = i_1$ there exists an $\varepsilon > 0$, so that we have

$$g_{i_1 i_2} \cdot (1+\varepsilon) \cdot \ldots \cdot g_{i_{k-1} ik} \cdot (1+\varepsilon) < 1$$

or equivalently

$$g_{i_1 i_2} \cdot \ldots \cdot g_{i_{k-1} ik} < 1,$$

may just not work.

In this chapter we will develop a stability test for interconnections of IOS/ISS systems that is independent of the structure of interconnections and does not rely on any step by step reduction of the interconnection graph. Instead, our results are based on on the developments of Chapter 2.

It is known that stability in the sense of Lyapunov (that is the ε-δ definition we gave in Section 1.4) for an equilibrium point with respect to a given dynamical system is equivalent to the existence of a Lyapunov function, asymptotic stability in turn is equivalent to the existence of a strict

Lyapunov function, and input-to-state stability of a system is equivalent to the existence of an ISS Lyapunov function [SW00].

The knowledge of a Lyapunov function makes it possible to check stability properties of an equilibrium point or a system, without actually computing any solution of the system. Moreover a Lyapunov function reveals information about the region of attraction of asymptotically stable equilibriums.

Unfortunately, as nice as Lyapunov functions are, so hard they are to find, at least in general. For some special cases, including linear time-invariant continuous-time systems, it is known how to construct Lyapunov functions. But most of the time, when we deal with nonlinear systems, only few construction methods for Lyapunov functions are available, even if one already knows that the system or equilibrium set satisfies certain stability properties. Therefore construction methods for Lyapunov functions for nonlinear and especially nonlinear large-scale systems are of big interest.

Here we will derive a procedure, how ISS Lyapunov functions of individual ISS systems can be used to construct an ISS Lyapunov function for the entire network. This might allow, e.g., to construct ISS Lyapunov functions for very high-dimensional systems, by constructing first ISS Lyapunov functions for individual lower-dimensional component systems and then to combine them into a Lyapunov function for the compound system.

Throughout this chapter we assume that norms are p-norms for one fixed $p \in [1, \infty]$, that is the same for all vector spaces, irrespective of their dimensions.

3.1. Stability concepts

First of all let us introduce several stability concepts. We consider systems with input signals u, states x, and outputs y of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \quad t \in \mathbb{R}_+, \tag{3.3}$$

where $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ is assumed to be continuous and *locally Lipschitz continuous in x uniformly for u in compacts*; by that we mean for each compact subset K_1 of $\mathbb{R}^N$ and every compact set K_2 of $\mathbb{R}^M$ there is some constant c so that $\|f(x,u) - f(z,u)\| \leq c\|x - z\|$ for all $x, z \in K_1$ and all $u \in K_2$. The function $h : \mathbb{R}^N \to \mathbb{R}^P$ is assumed to be continuous. Furthermore we require $f(0,0) = 0$ and $h(0) = 0$. We stick to the convenient convention to denote by small letters x, u, and y variables in $\mathbb{R}^N$, $\mathbb{R}^M$, and $\mathbb{R}^P$, and also signals, i.e., measurable and locally essentially bounded functions.

Solutions to (3.3) starting in x^0 at time t_0 with input $u \in L_\infty$ are generally denoted by $x(t, x^0, t_0, u)$ for $t \geq t_0$. Since the system is time invariant, we

may omit the reference to the initial time and assume it is zero. Recall that a system is said to be *forward complete* if for every initial state x^0 and input $u \in L_\infty$ the solution $x(t, x^0, u)$ is defined for all $t \geq 0$. By $y(t, x^0, u)$ we denote $h(x(t, x^0, u))$.

For a function $v : \mathbb{R} \to \mathbb{R}^N$ and $0 \leq t_1 \leq t_2$ by $v_{[t_1,t_2]}$ we denote the restriction of x to the interval $[t_1, t_2]$, i.e., the function given by

$$v_{[t_1,t_2]}(t) = \begin{cases} v(t) & \text{if } t \in [t_1, t_2], \\ 0 & \text{else.} \end{cases}$$

3.1.1. Trajectory estimates based stability concepts

Now we introduce a couple of stability concepts that will be further investigated in this chapter.

Definition 3.1.1 *A forward complete system* (3.3) *is* input-to-output practically stable *(IOpS for short), if there exists a $\mathcal{KL}$ function β, a $\mathcal{K}_\infty$ function γ, and $d \in \mathbb{R}_+$ such that*

$$\|y(t, x^0, t_0, u)\| \leq \beta(\|x^0\|, t - t_0) + \gamma(\|u\|_\infty) + d, \quad \forall t \geq t_0 \geq 0. \tag{3.4}$$

If $d = 0$ then the system is called input-to-output stable *(IOS). If $y = x$ in* (3.4) *then the system is* input-to-state practically stable *(ISpS), and if in addition $d = 0$ then it is called* input-to-state stable *(ISS), respectively.*

The term $\gamma(\|u\|_\infty)$ can be replaced by the norm of the restriction to past inputs $\gamma(\|u_{[t_0,t]}\|_\infty)$ and the sum can be replaced by a maximum or more generally, by a monotone aggregation function.

Input-to-state stability directly implies the following concepts:

Definition 3.1.2 *A forward complete system* (3.3) *has the* asymptotic gain property *(AG), if there exists $\gamma \in \mathcal{K}_\infty$, such that for all initial states $x^0 \in \mathbb{R}^N$, initial times $t_0 \geq 0$, and all $u(\cdot) \in L^\infty([0, \infty); \mathbb{R}^M)$ the estimate*

$$\limsup_{t\to\infty} \|x(t, x^0, t_0, u)\| \leq \gamma(||u||_\infty) \tag{3.5}$$

holds.

The asymptotic gain property states, that every trajectory must ultimately stay not far from zero, depending on the magnitude of $\|u\|_\infty$. Note that (3.5) is equivalent to

$$\begin{aligned} \limsup_{t\to\infty} \|x(t, x^0, t_0, u)\| &\leq \gamma(\text{ess.} \limsup_{t\to\infty} \|u(t)\|) \\ &= \text{ess.} \limsup_{t\to\infty} \gamma(\|u(t)\|), \end{aligned} \tag{3.6}$$

see [SW96]. Another stability property implied by ISS is global stability:

Definition 3.1.3 *A forward complete system* (3.3) *has the* global stability (GS) *property, if there exist* $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$*, such that for all initial states* $x^0 \in \mathbb{R}^N$*, all* $t \geq t_0 \geq 0$*, and all* $u(\cdot) \in L^\infty([0,\infty);\mathbb{R}^M)$ *the estimate*

$$\|x(t, x^0, t_0, u)\| \leq \sigma_1(\|x^0\|) + \sigma_2(||u||_\infty) \tag{3.7}$$

holds.

In the definitions of AG and GS we could replace the sum by a monotone aggregation function and qualitatively still have the same property.

All these stability concepts have a local version, namely, if the stability property only holds for x^0 and u in some open neighborhood of the origin. The local versions will be indicated by affixing an L to the stability name, like LISS. The only exception is LS for local stability instead of global stability.

For later reference the following equivalence relation is important:

Theorem 3.1.4 (Sontag,Wang) *A system* (3.3) *is ISS if and only it is AG and GS.*

For a proof see [SW96, Theorem 1].

Yet another stability concept that is obviously weaker than ISS is the 0-*global aymptotic stability property* (0-GAS): There exists a function $\beta \in \mathcal{KL}$ such that for the zero input, all $x^0 \in \mathbb{R}^N$, and all $t \geq t_0$,

$$\|x(t, x^0, t_0, \underline{0})\| \leq \beta(\|x^0\|, t). \tag{3.8}$$

In [SW96] it has been shown that 0-GAS implies LISS. In fact, to get LISS a weaker requirement is sufficient, which could be called 0-AS or 0-LAS: Local asymptotic stability of the origin for zero input. The proof is exactly the same as in [SW96, Lemma I.2]. A suitable converse Lyapunov theorem for local asymptotic stability can be found in [Mas56, Theorem 14*]. For future reference we put this observation into a lemma.

Lemma 3.1.5 *Consider the system* (3.3) *with* $x = y$*. If the origin is asymptotically stable (0-AS) for the unforced system*

$$\dot{x} = f(x, \underline{0}),$$

then system (3.3) *is LISS.*

3.1.2. Lyapunov function based concepts

Let us consider the Lyapunov formulation of the previously considered stability concepts. Recall that the derivative of a locally Lipschitz continuous function exists almost everywhere, cf. the Appendix.

Definition 3.1.6 *With respect to the system* (3.3) *a locally Lipschitz continuous function* $V : \mathbb{R}^N \to \mathbb{R}_+$ *is an* IOpS Lyapunov function *if there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that*

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^N, \tag{3.9}$$

and there exist $\gamma \in \mathcal{K}_\infty$, $d \in \mathbb{R}_+$, *and a positive* $\mathcal{KL}$ *function* α_3, *such that*

$$V(x) \geq \gamma(\|u\|) + d \implies \nabla V(x) f(x,u) \leq -\alpha_3(V(x), \|x\|), \quad \forall u \in \mathbb{R}^M \tag{3.10}$$

and for almost every $x \in \mathbb{R}^N$. *In the case that* $h = id$ *and* $d = 0$ *the function* V *is called* ISS Lyapunov function.

Later on (that is, in Lemma 3.3.17) we will see that we may replace (3.10) by the implication that for almost every $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}^M$

$$\begin{aligned} V(x) \geq \gamma(\|u\|) + d \text{ and } V(x) \neq 0 \\ \implies \nabla V(x) f(x,u) < 0. \end{aligned} \tag{3.11}$$

Remark 3.1.7 *In* [SW00] *for the preceding definition it seems that the function* α_3 *is taken only to be of class* $\mathcal{KL}$. *However, in* [SW00, Lemma A.2] *it becomes clear that the authors implicitly assume that* $\mathcal{KL}$ *functions are positive, so that we have added that assumption.*

Note that a general $\mathcal{KL}$ *function* α_3 *is clearly not sufficient in the defining implication of an IOpS Lyapunov function, since* $\alpha_3(V(x), \|x\|)$ *might be zero if* $\|x\|$ *is large. For example, if there exists a* $r^* > 0$ *such that* $\alpha_3(s,r) = 0$ *for all* s *and all* $r > r^*$, *then it will be impossible to derive* (3.4) *from* (3.10), *when the right hand side in* (3.10) *is zero for* $\|x\| > r^*$.

In [SW00] it has been proved that if a system is GS, then it is IOS if and only if it admits a smooth IOS Lyapunov function. It is well known, see [SW96], that the existence of a smooth ISS Lyapunov function is equivalent to the system being ISS in the sense of Definition 3.1.1. The equivalence for the practical ISS case has been argued in [JMW96] and is proved in [SW95]. The argument for nonsmooth Lyapunov functions goes along the lines of the same proofs, but it is not in the literature. We believe that a system satisfying GS is IOpS if and only if it admits an IOpS Lyapunov function. It should be possible to prove this similarly as in the other cases. As in Section 3.1.1 we define the corresponding local versions of the above Lyapunov functions on neighborhoods of the origin.

3.2. Problem statement

Consider n interconnected control systems given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, y_2 \ldots, y_n, u), \quad y_1 = h_1(x_1), \\ &\vdots \\ \dot{x}_n &= f_n(y_1, \ldots, y_{n-1}, x_n, u), \quad y_n = h_n(x_n), \end{aligned} \tag{3.12}$$

where $y_i \in \mathbb{R}^{P_i}, x_i \in \mathbb{R}^{N_i}, u \in \mathbb{R}^M$, and for each $i = 1, \ldots, n$ the functions $h_i : \mathbb{R}^{N_i} \to \mathbb{R}^{P_i}$ and $f_i : \mathbb{R}^{N_i + \sum_{j \neq i} P_j + M} \to \mathbb{R}^{N_i}$ are supposed to be locally Lipschitz continuous in x_i and y_j, $j \neq i$, uniformly for u in compact sets. We assume $f_i(0) = 0$ and $h_i(0) = 0$, $i = 1, \ldots, n$. Without loss of generality we may assume to have the same input u for all systems, since we may consider u as partitioned $u = (u_1^\top, \ldots, u_n^\top)$, such that each u_i is the input for subsystem i only. Then each f_i is of the form $f_i(\ldots, u) = \tilde{f}_i(\ldots, \pi_i(u)) = \tilde{f}_i(\ldots, u_i)$ with some projection π_i.

Assuming each of the subsystems of (3.12) to be stable in the same certain sense (i.e., one of IOpS/AG/GS, but the same concept for all subsystems), we are interested in conditions guaranteeing that the composite system given by $x = (x_1^\top, \ldots, x_n^\top)^\top$, $y = (y_1^\top, \ldots, y_n^\top)^\top$, $h = (h_1^\top, \ldots, h_n^\top)^\top$, and

$$\dot{x} = f(x, u) := \begin{pmatrix} f_1(x_1, h_2(x_2) \ldots, h_n(x_n), u) \\ \vdots \\ f_n(h_1(x_1), \ldots, h_n(x_{n-1}), x_n, u) \end{pmatrix}, \quad y = h(x), \tag{3.13}$$

is stable in the same sense (e.g., ISS from u to x). To this end we first have to say what we mean by "a subsystem is stable in a certain sense". Then we extend the definition of IOpS Lyapunov functions to interconnected systems and ask for a constructive method for a IOpS Lyapunov function for (3.13), based on the IOpS Lyapunov functions for the subsystems in (3.12)

3.2.1. IOpS estimates for the subsystems

Let us start with two examples to demonstrate that in the following it might be useful to not only consider sums of gains in the stability estimates in case of several inputs.

Example 3.2.1 *1. Consider the linear system*

$$\dot{x}_1 = Ax_1 + B_{12}x_2 + B_{13}x_3 \tag{3.14}$$

with $x_i \in \mathbb{R}^{N_i}$, for $i = 1, 2, 3$, $A \in \mathbb{R}^{N_1 \times N_1}$, and $B_{1i} \in \mathbb{R}^{N_1 \times N_i}$ for $i = 2, 3$. Assume that A is Hurwitz. Then for $x_2, x_3 = 0$ the origin of

the x_1-system is globally asymptotically stable. Now we are interested in bounds on the state for the case that x_2, x_3 are nonzero inputs.

Since A is Hurwitz, there exists positive constants $M, \lambda > 0$ such that $\|e^{At}\| \leq Me^{-\lambda t}$. By variation of the constants we obtain

$$\begin{aligned}\|x_1(t)\| &= \left\| e^{At}x_1(0) + \int_0^t e^{A(t-s)}(B_{12}x_2(s) + B_{13}x_3(s))ds \right\| \\ &\leq \|x_1(0)\|Me^{-\lambda t} + M\frac{1}{\lambda}\|B_{12}\|\|x_{2,[0,t]}\| + M\frac{1}{\lambda}\|B_{13}\|\|x_{3,[0,t]}\|\end{aligned}$$

Now $\beta(s,t) := s \cdot Me^{-\lambda t}$ defines a $\mathcal{KL}$ function and $\gamma_{1i}(s) = M\frac{1}{\lambda}\|B_{1i}\|s$ is of class $\mathcal{K}_\infty$ for $i = 2, 3$. So we have the ISS estimate

$$\|x_1(t)\| \leq \beta(\|x_1(0)\|, t) + \gamma_{12}(\|x_{2,[0,t]}\|) + \gamma_{13}(\|x_{3,[0,t]}\|)$$

and the linear system (3.14) *is input-to-state stable.*

2. *In addition to the first part of the example, we now assume that we are also given an output function y_1. Let $c^\top \in \{0,1\}^{1\times N_1}$, $c^\top \neq 0$. Let this vector encode the state variables, that we 'see' in the output: For some locally Lipschitz continuous $\mathcal{K}_\infty$ function κ let the output function be given by $y_1 = \kappa(|c^\top x_1(t)|^2)$. This yields the IOS estimate*

$$\|y_1(t)\| \leq \kappa\big(\beta(\|x_1(0)\|, t) + \gamma_{12}(\|x_{2,[0,t]}\|) + \gamma_{13}(\|x_{3,[0,t]}\|)\big).$$

Note that $\mu : (a_1, a_2, a_3) \mapsto \kappa(a + b + c)$ for $(a_1, a_2, a_3) \in \mathbb{R}_+^{1\times 3}$ gives a function $\mu \in \mathrm{MAF}_3$.

The first example showed that in the context of linear systems an aggregation of input gains as a sum seems to be natural. Now let us consider nonlinear systems. To derive ISS bounds for nonlinear systems the use of Lyapunov functions is often more natural than an approach depending on explicit solutions.

Example 3.2.2 *Consider the system*

$$\dot{x} = -x - 2x^3 + \frac{1}{2}(1 + 2x^2)u^2 + \frac{1}{2}y \tag{3.15}$$

where $x, y, u \in \mathbb{R}$. Take $V(x) = \frac{1}{2}x^2$ as a Lyapunov function candidate. We have

$$\dot{V} = -x^2 - 2x^4 + \frac{1}{2}x(1 + 2x^2)u^2 + \frac{1}{2}xy$$

along the trajectories of (3.15). Now if $|x| \geq u^2$ *and* $|x| \geq |y|$ *then this can be estimated as*

$$\dot{V} \leq -x^2 - 2x^4 + \frac{1}{2}x^2(1+2x^2) + \frac{1}{2}x^2 = -x^4 < 0$$

if $x \neq 0$. *The conditions* $|x| \geq u^2$ *and* $|x| \geq |y|$ *translate into* $|x| \geq \max\{u^2, |y|\}$ *and in terms of* V *this becomes*

$$V(x) \geq \max\{\sqrt{2} \cdot |u|, \sqrt{2|y|}\} \implies \dot{V}(x) \leq -x^4.$$

This is a Lyapunv ISS estimate where the gains are aggregated using a maximum.

Trajectory based estimates for the subsystems

Reconsider systems (3.12) and (3.13) with $h_i = \mathrm{id}$. If we consider individual systems, we treat $x_j, j \neq i$, as an independent input for system i. Thus, we call the i-th subsystem of (3.12) ISS, if there exist functions $\tilde{\beta}_i$ of class $\mathcal{KL}$, and $\gamma_{ij}, \tilde{\gamma}_i \in \mathcal{G}$, with $\gamma_{ii} = 0$, such that the solution $x_i(t)$ starting at time $t_0 \geq$ in $x_i(t_0)$ satisfies

$$\|x_i(t)\| \leq \tilde{\beta}_i(\|x_i(t_0)\|, t - t_0) + \sum_{j=1}^{n} \gamma_{ij}(\|x_{j\,[t_0,t]}\|_\infty) + \tilde{\gamma}_i(\|u\|_\infty) \tag{3.16}$$

for all $t \geq t_0$. We write $\Gamma^{ISS} = (\gamma_{ij})$. The functions γ_{ij} and $\tilde{\gamma}_i$ are called (nonlinear) *gains*. With $\tilde{\mu}_i \in \mathrm{MAF}_{n+2}$ we could also require

$$\|x_i(t)\| \leq \tilde{\mu}_i(\tilde{\beta}_i(\|x_i(t_0)\|, t - t_0), \gamma_{i1}(\|x_{1\,[t_0,t]}\|_\infty), \ldots, \gamma_{in}(\|x_{n\,[t_0,t]}\|_\infty), \tilde{\gamma}_i(\|u\|_\infty))$$

for all $t \geq t_0$, where we would assume that $\gamma_{ii} = 0$. We find that μ_i defined by $\mu_i(s) := \tilde{\mu}_i(0, s_1, \ldots, s_n, 0, 0)$ for $s \in \mathbb{R}^n_+$ belongs to MAF_n and that for $\beta_i := \tilde{\mu}_i(\tilde{\beta}_i, 0, \ldots) \in \mathcal{KL}$ and $\gamma_i = \tilde{\mu}_i(0, \ldots, 0, \tilde{\gamma}_i) \in \mathcal{G}$ we have, using subadditivity for monotone aggregation functions, that

$$\begin{aligned}\|x_i(t)\| \leq \beta_i(\|x_i(t_0)\|, t - t_0) + \\ \mu_i\big(\gamma_{i1}(\|x_{1\,[t_0,t]}\|_\infty), \ldots, \gamma_{in}(\|x_{n\,[t_0,t]}\|_\infty)\big) + \gamma_i(\|u\|_\infty)\end{aligned} \tag{3.17}$$

for all $t \geq t_0$. Here we note that we have $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{G}$ for $i = 1, \ldots, n$.

It will turn out that for most of our needs it suffices to consider $\mu_i \in \mathrm{MAF}_n$ and apply this to the gains that correspond to inputs that come from inside

the network, i.e., from other subsystems. Note that in the case $\tilde{\mu}_i = \oplus$ then instead of (3.17) we could write

$$\|x_i(t)\| \leq \beta_i(\|x_i(t_0)\|, t - t_0) \oplus \bigoplus_{j=1}^{n} \gamma_{ij}(\|x_{j\,[t_0,t]}\|_\infty) \oplus \gamma_i(\|u\|_\infty) \qquad (3.17')$$

for $t \geq t_0$, where in this case $\beta_i = \tilde{\beta}_i$ and $\gamma_i = \tilde{\gamma}_i$, $i = 1, \ldots, n$.

We also need GS and AG definitions for the subsystems: Subsystem i of (3.12) with $h_j = \mathrm{id}$ for all j is AG, if there exist $\gamma_{ij}, \gamma_i, \in \mathcal{G}$, $\gamma_{ii} = 0$, $\mu_i \in \mathrm{MAF}_n$, $i, j = 1, \ldots, n$, such that

$$\limsup_{t\to\infty} \|x_i(t, x_i^0, t_0; x_j(\cdot), j \neq i, u(\cdot))\| \leq \mu_i\big(\gamma_{i1}(\|x_1\|_\infty), \ldots, \gamma_{in}(\|x_n\|_\infty)\big) + \gamma_i(||u||_\infty), \qquad (3.18)$$

and it is GS, if there exist $\gamma_{1i}, \sigma_{2ij}, \sigma_{3i} \in \mathcal{G}$, $\sigma_{2ii} = 0$, $\mu_i \in \mathrm{MAF}_n$, $i, j = 1, \ldots, n$, such that

$$\|x_i(t, x_i^0, t_0; x_j(\cdot), j \neq i, u(\cdot))\| \leq \sigma_{1i}(\|x_i^0\|) + \mu_i\big(\sigma_{2i1}(||x_1||_\infty), \ldots, \sigma_{2in}(||x_n||_\infty)\big) + \sigma_{3i}(||u||_\infty), \qquad (3.19)$$

where both (3.18) and (3.19) are supposed to hold for all $x_i^0 \in \mathbb{R}^{N_i}$; $t_0 \geq 0$; $u(\cdot) \in L^\infty([0,\infty); \mathbb{R}^M)$ and $x_j(\cdot) \in L^\infty([0,\infty); \mathbb{R}^{N_j})$ for all $j \neq i$ and all $t \geq t_0$, for $i = 1, \ldots, n$. Again, according to [SW96], (3.18) can equivalently be stated as

$$\limsup_{t\to\infty} \|x_i(t, x_i^0, t_0; x_j(\cdot), j \neq i, u(\cdot))\| \leq \operatorname{ess.}\limsup_{t\to\infty}\ \mu_i\big(\gamma_{i1}(\|x_1(t)\|), \ldots, \gamma_{in}(\|x_n(t)\|)\big) + \operatorname{ess.}\limsup_{t\to\infty}\ \gamma_i(\|u(t)\|). \qquad (3.18')$$

We denote $\Gamma^{AG} = (\gamma_{ij})_{i,j=1}^n$ and $\Gamma^{GS} = (\sigma_{2ij})_{i,j=1}^n$. The matrices Γ^{ISS}, Γ^{AG}, and Γ^{GS} are referred to as *gain matrices*.

For vector functions $x = (x_1^\top, \ldots, x_n^\top)^\top : \mathbb{R}_+ \to \mathbb{R}^{N_1+\ldots+N_n}$ and $y = (y_1^\top, \ldots, y_n^\top)^\top : \mathbb{R}_+ \to \mathbb{R}^{P_1+\ldots+P_n}$ such that $x_i : \mathbb{R}_+ \to \mathbb{R}^{N_i}$, $y_i : \mathbb{R}_+ \to \mathbb{R}^{P_i}$, $i = 1, \ldots, n$, and times $0 \leq t_1 \leq t_2$, $t \geq 0$ we define

$$\left\| x_{[t_1,t_2]} \right\| := \begin{pmatrix} \|x_{1,[t_1,t_2]}\|_\infty \\ \vdots \\ \|x_{n,[t_1,t_2]}\|_\infty \end{pmatrix} \in \mathbb{R}^n_+, \quad \text{and} \quad \left| x(t) \right| := \begin{pmatrix} \|x_1(t)\| \\ \vdots \\ \|x_n(t)\| \end{pmatrix} \in \mathbb{R}^n_+ .$$

For $u(\cdot) \in L^\infty([0,\infty);\mathbb{R}^M)$, finite times $t \geq 0$, and for $s \in \mathbb{R}^n_+$ we introduce the abbreviating notation

$$\underline{\gamma}(s) := \begin{pmatrix} \gamma_1(s_1) \\ \vdots \\ \gamma_n(s_n) \end{pmatrix} \in \mathbb{R}^n_+ \quad \text{and} \quad \underline{\beta}(s,t) := \begin{pmatrix} \beta_1(s_1,t) \\ \vdots \\ \beta_n(s_n,t) \end{pmatrix}, \tag{3.20}$$

where $\gamma_1, \ldots, \gamma_n$ come from (3.17) or (3.18) and β_i from (3.17). Further we define the vector $\underline{e} = (1,\ldots,1)^\top$ of ones.

Now we can rewrite the ISS conditions (3.17) of the subsystems in a vectorized form for $t \geq t_0$ as

$$\left|\!\left| x(t) \right|\!\right| \leq \underline{\beta}(\left|\!\left| x(t_0) \right|\!\right|, t-t_0) + \Gamma^{ISS}_\mu(\left\|\!\left\| x_{[t_0,t]} \right\|\!\right\|) + \underline{\gamma}(\|u\|_\infty \cdot \underline{e}). \tag{3.21}$$

The AG conditions now read

$$\limsup_{t\to\infty} \left|\!\left| x(t) \right|\!\right| \leq \Gamma^{AG}_\mu(\left\|\!\left\| x_{[t_0,\infty]} \right\|\!\right\|) + \underline{\gamma}(\|u\|_\infty \cdot \underline{e}). \tag{3.22}$$

For the GS assumption we obtain

$$\left|\!\left| x(t) \right|\!\right| \leq \underline{\sigma_1}(\left|\!\left| x^0 \right|\!\right|) + \Gamma^{GS}_\mu(\left\|\!\left\| x_{[t_0,t]} \right\|\!\right\|) + \underline{\gamma_3}(\|u\|_\infty \cdot \underline{e}), \tag{3.23}$$

where $\underline{\sigma_1}$ and $\underline{\gamma_3}$ are defined analogously to $\underline{\gamma}$, with respect to σ_{1i} and γ_{3i}, $i = 1,\ldots,n$.

We assume in general that supremum or essential-supremum norms for functions are induced by the norms on the spaces where the functions take their values in. We state some easy-to-see properties of our norm-notation as remark, for easy reference.

Remark 3.2.3 *In this chapter we assume that all norms are p-norms for one fixed $p \in [1,\infty]$, this p being the same for all vector spaces, irrespective of their dimensions. For a fixed p the p-norms on the spaces $\mathbb{R}^n$, $\mathbb{R}^{N_i}$, $i = 1,\ldots,n$, and $\mathbb{R}^N$, where $N = \sum_{i=1}^n N_i$, are compatible in the way that we have for $x \in \mathbb{R}^N$*

$$\|\left|\!\left| x \right|\!\right|\| = \|x\|, \tag{3.24}$$

as a straight-forward calculation shows.

In principle we could use arbitrary monotone norms on each vector space, but would have to replace (3.24) *with two inequalities, coming from the equivalence of norms on finite dimensional Euclidean spaces. This wouldn't fundamentally change the following results. Essential is that the norm on $\mathbb{R}^n_+$ is a monotone one. For clarity we content ourselves with p-norms.*

For any continuous function $s : \mathbb{R}_+ \to \mathbb{R}^n_+$ and any $0 \le a \le b$ we have for any choice of norms on the respective vectorspaces

$$\| s_{[a,b]} \| = \sup_{t\in[a,b]} | s(t) | .$$

But we ***do not have*** *that for all essentially bounded functions $x : \mathbb{R}_+ \to \mathbb{R}^N$*

$$\|x_{[a,b]}\|_\infty = \| \| x_{[a,b]} \| \|,$$

consider, e.g.,

$$1 = \sup\left\{ \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_1, \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_1 \right\} \neq \left\| \sup\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right\|_1 = 2.$$

But for monotone norms on $\mathbb{R}^N$ we have at least an inequality,

$$\| \| x_{[a,b]} \| \| = \left\| \begin{pmatrix} \|x_{1,[a,b]}\|_\infty \\ \vdots \\ \|x_{n,[a,b]}\|_\infty \end{pmatrix} \right\| \le \big\| \|x_{[a,b]}\|_\infty \cdot \underline{e} \big\| = \|\underline{e}\| \cdot \|x_{[a,b]}\|_\infty. \quad (3.25)$$

In particular, estimate (3.25) holds for all p-norms and the max-norm on $\mathbb{R}^N$, cf. Subsection 1.2.2.

Lyapunov function based estimates for the subsystems

Assume that for for $i = 1, \dots, n$ there exist locally Lipschitz continuous IOpS Lyapunov functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and functions $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$, $\gamma_{ij} \in \mathcal{G}$ with $\gamma_{ii} = 0$, monotone aggreation functions $\mu_i \in \mathrm{MAF}_n$, $\gamma_i \in \mathcal{K}_\infty$, positive $\mathcal{KL}$ functions α_i, and real numbers $d_i \ge 0$, such that

$$\alpha_{1i}(\|h_i(x_i)\|) \le V_i(x_i) \le \alpha_{2i}(\|x_i\|) \quad (3.26)$$

for all $x_i \in \mathbb{R}^{N_i}$, and

$$\begin{aligned} V_i(x_i) \ge \mu_i\big(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n))\big) + \gamma_i(\|u\|) + d_i \\ \Longrightarrow \nabla V_i(x_i) \cdot f_i(x,u) \le -\alpha_i(V_i(x_i), \|x_i\|) \end{aligned} \quad (3.27)$$

for almost all $x_i \in \mathbb{R}^{N_i}$. The functions γ_{ij} are called *IOpS Lyapunov gains* or simply *gains*, if no confusion arises. We write $\Gamma = (\gamma_{ij})$ and $\mu = (\mu_1, \dots, \mu_n)^\top \in \mathrm{MAF}_n^n$. Note that the assumption $\gamma_{ii} = 0$ is important for the implication (3.27), otherwise the implication would have a different meaning.

In the case of $\mu = \oplus$ instead of (3.27) we would write

$$\begin{aligned} V_i(x_i) \geq \bigoplus_{j=1}^{n} \gamma_{ij}(V_j(x_j)) \oplus \gamma_i(\|u\|) \oplus d_i \\ \Longrightarrow \nabla V_i(x_i) \cdot f_i(x,u) \leq -\alpha_i(V_i(x_i), \|x_i\|). \end{aligned} \tag{3.27$'$}$$

In the general case when $y_i = h_i(x_i)$ and feedback interconnections are to be considered, we have to make an additional assumption for the solutions of interconnected system to exist. This assumption is the *unboundedness observability property* (UO) and states that there exist $\alpha_i^0 \in \mathcal{K}_\infty$ and constants $C_i \geq 0$ such that solutions of the individual systems (3.12) satisfy

$$\|x_i(t)\| \leq \alpha_i^0\Big(\|x_i^0\| + \sum_{j=1}^{n} \|y_{j,[t_0,t]}\| + \|u_{[t_0,t]}\|\Big) + C_i \tag{3.28}$$

for all $t \geq t_0$ and $i = 1, \ldots, n$. It is important to remark that the sum in (3.28) incorporates the term $\|y_{i,[t_0,t]}\|$. Note that in (3.28) instead of the sum we could equivalently place any monotone aggregation function. But since all these formulations are qualitatively equivalentso we content ourselves with one formulation.

Another useful fact is, that if a system is ISpS then it automatically satisfies UO, as is immediate from (3.17) for $t = t_0$: Using the weak triangular inequality repeatedly, as well as subadditivity of monotone aggregation functions, there clearly exists a function α_i^0 of class $\mathcal{K}_\infty$ so that the right-hand side of (3.17) is less than the right-hand side of (3.28), since it does not incorporate the term $\|y_{i,[t_0,t]}\|$.

3.3. Small gain theorems

In this section we state and prove small gain theorems for several stability notions. These are AG, GS, ISS for the trajectory based estimates, and IOpS for the Lyapunov estimates. From the Lyapunov small gain theorem we derive IOS, ISpS, and ISS small gain theorems. We begin with the trajectory based concepts.

Let $\mu \in \mathrm{MAF}_n^n$. Given a gain matrix Γ, we say that Γ_μ satisfies the *small gain condition*, if there exists an operator $D = \mathrm{diag}(\mathrm{id} + \alpha)$ for some $\alpha \in \mathcal{K}_\infty$ such that for all $s \in \mathbb{R}_+^n$, $s \neq 0$ we have

$$\Gamma_\mu \circ D(s) := \Gamma_\mu(D(s)) \not\geq s. \tag{3.29}$$

This condition can equivalently be stated as: There has to be at least one $i \in \{1, \dots, n\}$, such that the ith component of $\Gamma_\mu \circ D$ decreases, i.e., $\Gamma_\mu(D(s))_i < s_i$. In other words (3.29) states that there exists no $s > 0$, such that $\Gamma_\mu \circ D(s) \geq s$. Note that $\Gamma_\mu \circ D(s) \ngeq s$ for all $s \neq 0$ if and only if $D \circ \Gamma_\mu(s) \ngeq s$ for all $s \neq 0$, see Lemma 2.2.12.

3.3.1. AG, GS, and ISS

Here we consider the case when $y = x$ and $d = 0$, i.e., we assume outputs are equal to the state. Then from AG, GS, and IOpS we obtain stability concepts that are more familiar from the literature: The asymptotic gain (AG) and global stability (GS) properties, as well as input-to-state stability (ISS).

Theorem 3.3.1 *Assume each subsystem of* (3.12) *is GS. If* Γ_μ^{GS} *satisfies the small gain condition* (3.29)*, then* (3.13) *possesses the global stability property.*

Theorem 3.3.2 *Assume each subsystem of* (3.12) *is AG and solutions of the composite system* (3.13) *exist for all times and are essentially bounded. If* Γ_μ^{AG} *satisfies the small gain condition* (3.29)*, then* (3.13) *has the asymptotic gain property.*

Note that here we explicitly have to ask for the existence of trajectories for all times. This is in general not guaranteed by the AG condition, as Example 3.3.9, which is based on an example in [Hah67], demonstrates.

Before proving Theorems 3.3.1 and 3.3.2 we note some immediate consequences. Combining the two theorems we obtain the following obvious but important result.

Corollary 3.3.3 *Assume each subsystem of* (3.12) *is GS and AG. If both* Γ_μ^{AG} *and* Γ_μ^{GS} *satisfy the small gain condition* (3.29)*, then* (3.13) *has the global stability property and the asymptotic gain property.*

This result is particularly interesting, if the matrices Γ_μ^{AG} and Γ_μ^{GS} agree. Since AG and GS together are equivalent to ISS, see Theorem 3.1.4, this can be reformulated to obtain one of our main results:

Theorem 3.3.4 *Consider the system* (3.12) *and suppose that each subsystem is ISS, i.e., condition (3.17) holds for all* $i = 1, \dots, n$ *with some* $\mu \in \mathrm{MAF}_n^n$*. Let* Γ *be given by* (3.17)*. If there exists a mapping* $D = \mathrm{diag}(id + \alpha)$ *for some* $\alpha \in \mathcal{K}_\infty$ *such that*

$$(\Gamma_\mu \circ D)(s) \ngeq s$$

for all $s \neq 0$*, then the system (3.13) is ISS from* u *to* x*.*

Proof. The ISS condition (3.17) implies

$$\|x_i(t)\| \le \beta_i(\|x_i(t_0)\|,0)+ \\ \mu_i\big(\gamma_{i1}(\|x_1\|_\infty),\dots,\gamma_{in}(\|x_n\|_\infty)\big)+\gamma_i(\|u\|_\infty)$$

But this is just a GS estimate, implying that $\sigma_{2ij} \le \gamma_{ij}$ for the GS estimates σ_{2ij}, for $i,j = 1,\dots,n$. This gives $\Gamma_\mu \ge \Gamma_\mu^{GS}$ and makes Theorem 3.3.1 applicable. In particular this shows that solutions exist for all times and are uniformly bounded.

Now (3.17) clearly implies (3.18), i.e., ISS implies AG and we may assume $\Gamma_\mu \ge \Gamma_\mu^{AG}$ such that Theorem 3.3.2 is applicable. Together we obtain that the interconnection is GS and AG, hence it is ISS by Theorem 3.1.4. □

In the case that all ISS conditions for the subsystems are given in the maximum formulation (3.17′), we do not need the operator D, since we can apply Theorem 2.5.4, which does not require D, instead of Theorem 2.5.1 to solve monotone inequalities in the proof. This consideration leads to the following consequence of the above theorem:

Corollary 3.3.5 *Consider the system* (3.12) *and suppose that each subsystem is ISS so that condition* (3.17′) *holds, i.e., we have* $\mu_i = \oplus$ *for all* $i = 1,\dots,n$. *Let* Γ *be given by* (3.17′). *If*

$$\Gamma_\oplus(s) \not\ge s$$

for all $s \neq 0$, *or, equivalently, if every cycle in* Γ *is a contraction, then the system (3.13) is ISS from* u *to* x.

We postpone the proof of Corollary 3.3.5 until after the proofs of Theorems 3.3.1 and 3.3.2.

Remark 3.3.6 *For nonlinear gains and* $n = 2$ *the condition (3.29) in Theorem 3.3.4 reads as follows: There exists* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\Gamma_\mu \circ D(s) = \begin{pmatrix} \gamma_{12}\circ(id+\alpha)(s_2) \\ \gamma_{21}\circ(id+\alpha)(s_1) \end{pmatrix} \not\ge \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

for all $s = (s_1,s_2)^\top \in \mathbb{R}^2_+$. *This is easily seen to be equivalent to*

$$\gamma_{12}\circ(id+\alpha)\circ\gamma_{21}\circ(id+\alpha)(t) < t \tag{3.30}$$

for all $t > 0$. *To this end we consider the vector* $(\gamma_{12}\circ(id+\alpha)(t),t)^\top$ *for any* $t > 0$. *Applying* $\Gamma_\mu \circ D$ *we see that the first component does not change, hence the second one must be less than* t, *which is just* (3.30). *The converse implication is obvious. Now inequality* (3.30) *is equivalent to the condition*

in the small gain theorem of [JTP94]*, namely, that for some* $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ *it should hold that*

$$(id + \tilde{\alpha}_1) \circ \gamma_{21} \circ (id + \tilde{\alpha}_2) \circ \gamma_{12}(s) \leq s \tag{3.31}$$

for all $s \in \mathbb{R}_+$*, by taking* $\alpha_1 = \alpha_2 = \alpha$*, or conversely,*

$$\alpha(s) = \min\{\alpha_1(s), \alpha_2(s)\}.$$

Hence our theorem contains this result as a particular case.

The condition (3.31) of [JTP94] seems to be very similar to the small gain condition $\gamma_{12} \circ \gamma_{21}(s) < s$ of [JMW96] and [Grü02b], however in these papers the maximum formulation of ISS is used, so that the gains are different. This similarity raises the question, whether the compositions with $(\mathrm{Id} + \tilde{\alpha}_i)$, $i = 1, 2$ in (3.31) or more generally with D in (3.29) are necessary in the context of the summation formulation of ISS, or in any other more general formulation using monotone aggregation functions. The following example shows that this is indeed the case.

Example 3.3.7 *Consider the equation*

$$\dot{x} = -x + u(1 - e^{-u}), \quad x(0) = x^0 \in \mathbb{R},\ u \in \mathbb{R}.$$

Integrating for $t \geq 0$ *it follows that*

$$\begin{aligned} x(t) &= e^{-t}x^0 + \int_0^t e^{-(t-\tau)}u(\tau)(1 - e^{-u(\tau)})\, d\tau \\ &\leq e^{-t}x^0 + ||u||_\infty(1 - e^{-||u||_\infty}) = e^{-t}x^0 + \gamma(\|u\|_\infty), \end{aligned}$$

with $\gamma(s) = s(1 - e^{-s})$*. Clearly* $\gamma(s) < s$*. Then for a feedback system*

$$\dot{x}_1 = -x_1 + x_2(1 - e^{-x_2}) + u(t), \tag{3.32}$$

$$\dot{x}_2 = -x_2 + x_1(1 - e^{-x_1}) + u(t) \tag{3.33}$$

we have ISS for each subsystem with $x_i(t) \leq e^{-t}x_i^0 + \gamma_i(||x_i||_\infty) + \eta_i(||u||_\infty)$*, where* $\gamma_i(s) < s$ *and hence* $\gamma_1 \circ \gamma_2(s) < s$ *for* $s > 0$*, but there are solutions* $x_1 = x_2 = const$ *given by*

$$\dot{x}_1 = -x_2 e^{-x_2} + u, \quad with\ u = x_2 e^{-x_2}.$$

Here $x_1 = x_2$ *can be chosen arbitrary large with* $u \to 0$ *for* $x_1 \to \infty$*, so that the system cannot satisfy the asymptotic gain property and is therefore not ISS. Hence the condition* $\Gamma_\mu(s) \not\geq s$*, for all* $s \in \mathbb{R}^n_+ \setminus \{0\}$*, or for two*

subsystems $\gamma_{12} \circ \gamma_{21}(s) < s$, for all $s > 0$, is not sufficient for the input-to-state stability of the composite system in the general nonlinear case, except for $\mu = \oplus$. If $\mu = \oplus$ then $\Gamma_\oplus(s) \not< s$ for all $s \neq 0$ is indeed sufficient, but the gains γ would of course be different, i.e., larger, so that the small gain condition effectively does not hold.

We proceed with the proof of Theorems 3.3.1 and 3.3.2.

Proof of Theorem 3.3.1. We may assume that the vectorized formulation (3.23) of the GS conditions (3.19) for all $x^0 \in \mathbb{R}^{N_1+\ldots+N_n}$, and $u(\cdot) \in L^\infty(\mathbb{R}_+;\mathbb{R}^M)$ is satisfied for some $t > t_0$:

$$\left|\!\left| x(t) \right|\!\right| \leq \underline{\sigma_1}(\left|\!\left| x^0 \right|\!\right|) + \Gamma_\mu^{GS}(\left\|\!\left| x_{[t_0,t]} \right|\!\right\|) + \underline{\gamma_3}(\|u\|_\infty \cdot \underline{e}).$$

Taking the supremum on both sides over $\tau \in [t_0, t]$ we obtain by Remark 3.2.3 that

$$(\mathrm{id} - \Gamma_\mu^{GS}) \left(\left\|\!\left| x_{[t_0,t]} \right|\!\right\|\right) \leq \quad \underline{\sigma_1}(\left|\!\left| x^0 \right|\!\right|) + \underline{\gamma_3}(\|u\|_\infty \cdot \underline{e}).$$

The right hand side is independent of $t > t_0$, hence this estimate holds for all $t > t_0$. Now by Theorem 2.5.1 we find

$$\begin{aligned} \|\left\|\!\left| x_{[0,t]} \right|\!\right\|\| &\leq \varphi\left(\left\|\underline{\sigma_1}(\left|\!\left| x^0 \right|\!\right|) + \underline{\gamma_3}(\|u\|_\infty \cdot \underline{e})\right\|\right) \\ &\leq \varphi\left(2 \cdot \left\|\underline{\sigma_1}(\left|\!\left| x^0 \right|\!\right|)\right\|\right) + \varphi\left(2 \cdot \left\|\underline{\gamma_3}(\|u\|_\infty \cdot \underline{e})\right\|\right) \quad (3.34) \\ &=: s_\infty \quad (3.35) \end{aligned}$$

for some class $\mathcal{K}$ function φ and all times $t \geq t_0$. Hence for every initial condition and essentially bounded input u the solution of our system (3.13) exists for all times $t \geq t_0$ and is uniformly bounded, since s_∞ in (3.35) does not depend on t. The GS estimate for (3.13) is then given by (3.34). □

For the proof of the AG small gain theorem we need a short technical argument which is served as a lemma.

Lemma 3.3.8 *Let $s : \mathbb{R}_+ \to \mathbb{R}_+^n$ be continuous and bounded. Then*

$$\limsup_{t\to\infty} s(t) = \limsup_{t\to\infty} \left\|\!\left| s_{[t/2,\infty)} \right|\!\right\|.$$

Proof. Let $\limsup_{t\to\infty} s(t) =: a \in \mathbb{R}_+^n$ and $\limsup_{t\to\infty} \left\|\!\left| s_{[t/2,\infty)} \right|\!\right\| =: b \in \mathbb{R}_+^n$. For every $\varepsilon \in \mathbb{R}_+^n$, $\varepsilon \gg 0$, there exist $t_a, t_b \geq 0$ such that

$$\forall\, t \geq t_a : \sup_{t\geq t_a} s(t) \leq a + \varepsilon \quad \text{and} \quad \forall\, t \geq t_b : \sup_{t\geq t_b} \left\|\!\left| s_{[t/2,\infty)} \right|\!\right\| \leq b + \varepsilon. \quad (3.36)$$

Clearly we have

$$s(t) \leq \left\|\!\left| s_{[t/2,\infty)} \right|\!\right\|$$

for all $t \geq 0$, i.e., $a \leq b$. On the other hand $s(\tau) \leq a + \varepsilon$ for $\tau \geq t$ implies $\left\|\!\left| s_{[\tau/2,\infty)} \right|\!\right\| \leq a + \varepsilon$ for $\tau \geq 2t$, i.e., $b \leq a$. □

Proof of Theorem 3.3.2. Assume solutions of (3.13) with essentially bounded input exist for all times, are uniformly bounded, and the AG formulation (3.22) is valid, i.e., we have for all $t_0 \geq 0$, all initial conditions, and all inputs $u(\cdot) \in L^\infty(\mathbb{R}_+; \mathbb{R}^M)$ that

$$\limsup_{t\to\infty} \left| x(t) \right| \leq \Gamma_\mu^{AG}(\left\| x_{[t_0,\infty)} \right\|) + \underline{\gamma}(\|u\|_\infty \cdot \underline{e}). \tag{3.37}$$

By Lemma 3.3.8 we have

$$\limsup_{t\to\infty} \left| x(t) \right| = \limsup_{t_0\to\infty} \left\| x_{[t_0,\infty)} \right\| =: l(x) \in \mathbb{R}_+^n,$$

hence by (3.6) inequality (3.37) can be stated equivalently as

$$l(x) \leq \Gamma_\mu^{AG}(l(x)) + \underline{\gamma}(\|u\|_\infty \cdot \underline{e})$$

or

$$(\mathrm{id} - \Gamma_\mu^{AG}) \circ l(x) \leq \underline{\gamma}(\|u\|_\infty \cdot \underline{e}).$$

Finally, by Theorem 2.5.1 we have

$$\|l(x)\| \leq \varphi(\|\underline{\gamma}(\|u\|_\infty \cdot \underline{e})\|) \tag{3.38}$$

for some φ of class $\mathcal{K}_\infty$, which is the desired asymptotic gain property (3.5). □

Now we can state the proof of Corollary 3.3.5, which is a combination of slight modifications of the proofs of Theorems 3.3.1 and 3.3.2.

Proof of Corollary 3.3.5. By Lemma 2.3.14 the cycle condition and the no joint increase condition are equivalent for $\Gamma_\oplus$, i.e., in the case that $\mu = \oplus$.

The proof of the corollary comprises to prove that the network satisfies GS and AG. It goes along the the proofs of Theorems 3.3.1 and 3.3.2, with every sum and every monotone aggregation function replaced by a maximum and every application of Theorem 2.5.1 replaced by an application of Theorem 2.5.4, i.e., the corresponding tool to find bounds for monotone inequalities, but in the max-formulation. □

Finally we give the example that shows, that AG and the small gain condition alone are not sufficient for solutions of interconnected systems to exist for all times.

Example 3.3.9 *Consider the planar autonomous system defined by*

$$\dot{x} = f(x,y) := \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)} \tag{3.39}$$

$$\dot{y} = g(x,y) := \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)}. \tag{3.40}$$

It is shown in [Hah67, §40, pp 191–194] *that for* (3.39) *and* (3.40) *the origin is globally attractive but unstable.*

Now replace (3.40) *by*

$$\dot{y} = \tilde{g}(x,y,v) := \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)} \exp(v^2), \tag{3.41}$$

to obtain the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ \tilde{g}(x,y,v) \end{pmatrix}, \tag{3.42}$$

where $v \in \mathbb{R}$ is a parameter.

By the same methods as in [Hah67, §40, pp 191–194] *it can be shown that for any fixed $v \in \mathbb{R}$ the origin for system* (3.42) *is unstable, yet globally attracting. By symmetry it suffices to consider the open upper half plane in $\mathbb{R}^2$. The proof is exactly as in* [Hah67] *and therefore omitted.*

Next we show, that for $u(\cdot) \in L^\infty(\mathbb{R}_+;\mathbb{R})$, $\|u\|_\infty < m$, the origin of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ \tilde{g}(x,y,u) \end{pmatrix} \tag{3.43}$$

is globally attracting and unstable: Suppose a trajectory $\varphi(\cdot,(x^0,y^0)^\top,u(\cdot))$ of (3.43) *starts in $(x^0,y^0)^\top$ lying in*

$$\Psi^+ := \{(x,y)^\top \in \mathbb{R}^2 : (x \le 0 \text{ and } y > 0) \text{ or } (x > 0 \text{ and } y > 2x)\}.$$

In the region Ψ^+ the second component of the trajectory $\varphi(\cdot) := \varphi(\cdot,(x^0,y^0)^\top,u(\cdot))$ is bounded by the second component of the trajectory $\eta(\cdot) := \eta(\cdot,(x^0,y^0)^\top,||u||_\infty)$ of system (3.42) *in the sense that for all $t,\tau \ge 0$ such that $\varphi(t),\eta(\tau) \in \Psi^+$ we have*

$$\varphi(t)_1 = \eta(\tau)_1 \quad \Rightarrow \quad \varphi(t)_2 \le \eta(\tau)_2.$$

This follows as for almost all t we have $\tilde{g}(x,y,u(t)) \le \tilde{g}(x,y,\|u\|_\infty)$.

Now $\eta(\cdot)$ is known to reach the half line $H := \{(x,y)^\top \in \mathbb{R}^2 : x \geq 0, y = 2x\}$ in finite time (using the argument provided in [Hah67]*). Since in Ψ^+ we have $\dot{x} > 0$ and $\dot{y} > 0$, the trajectory $\varphi(\cdot,(x^0,y^0)^\top,u)$ also hits the line H in finite time $\tau > 0$ at the point $(x_\tau, y_\tau)^\top$.*

Define the "attracting set" $A := \{(x,y)^\top \in \mathbb{R}^2 : y \geq 0, x > \frac{1}{2}y\}$. Trajectories with bounded input starting on the line $H \subset \overline{A}$ are ultimately attracted to the origin: On H every velocity vector points into A, in A the y-component of each velocity vector is strictly negative. At some point each trajectory in A reaches a neighborhood of the x-axis, where all velocity vectors point to the lower left.

The preceding argument shows, that system (3.43) *has the asymptotic gain property, with arbitrary small gain γ. So with slight abuse of notation we may choose $\gamma = 0$ as the asymptotic gain.*

Now consider the feedback interconnection of two systems (3.43)*, each with input $v = y$ as depicted in Figure 3.2:*

$$\begin{aligned} \Sigma_1 : \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} &= \begin{pmatrix} f(x_1, y_1) \\ \tilde{g}(x_1, y_1, y_2) \end{pmatrix} \text{ and} \\ \Sigma_2 : \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} f(x_2, y_2) \\ \tilde{g}(x_2, y_2, y_1) \end{pmatrix} \end{aligned} \tag{3.44}$$

We concentrate only on the 'diagonal', that is, we let each system start at $(x^0,y^0)^\top \in \mathbb{R}^2$ and denote $x_1 = x_2 = x$ and $y_1 = y_2 = y$. Hence the evolution of both systems is exactly the same, motivating the term diagonal. The small gain condition, which in this case reduces to $\gamma^2 \equiv 0 < id$, is clearly satisfied. On the diagonal the dynamics are given by

$$\dot{x} = f(x,y)\,, \quad \dot{y} = \tilde{g}(x,y,y)\,. \tag{3.45}$$

For suitable initial conditions system (3.45) *has a finite escape time, as the next calculation shows: Let $V(x,y) = \frac{1}{2}y^2$ be a Lyapunov function. We will show that $\dot{V} > V^2$ along trajectories for suitable initial conditions. This implies that not all trajectories of the feedback system exist for all times.*

First we define the set

$$Q := \left\{ (x,y)^\top \in \mathbb{R}^2 : y > 0, 0 \leq x < \frac{y(e^y - y^2 - 1)}{2(e^y - 1)} \right\}.$$

On Q we have

$$\dot{x} < \frac{1}{2}\dot{y} \tag{3.46}$$

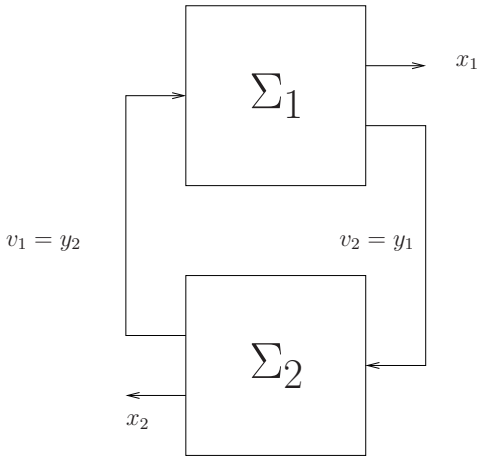


Figure 3.1.: Feedback system (3.44).

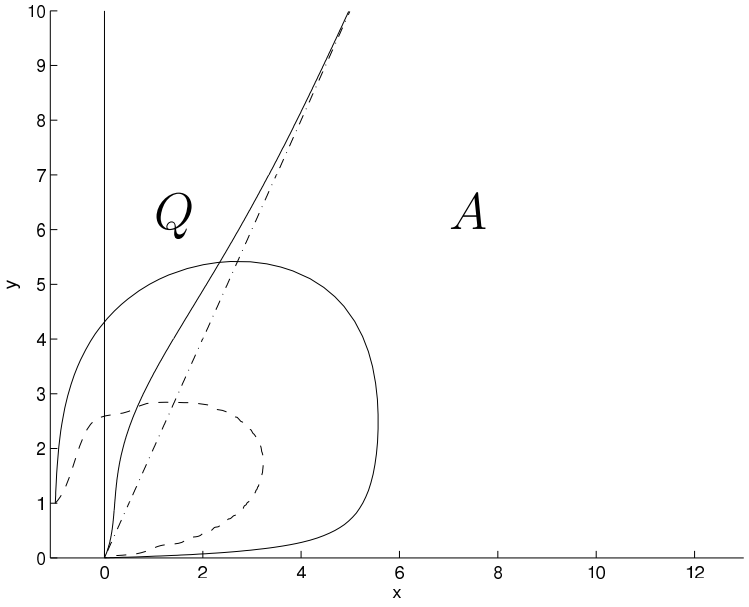


Figure 3.2.: A trajectory of the (3.41) in Example 3.2: The dashed trajectory starts in $(-1, 1)^\top$ with input $\sqrt{3}\sin(t)$. The solid trajectory starts in the same point with constant input equal to $\sqrt{3}$. The half line H with (slope 2) is dashed dotted. The regions Q and A are depicted as well.

for system (3.44). *Asymptotically we have* $\frac{y(e^y-y^2-1)}{2(e^y-1)} \approx \frac{1}{2}y$ *as* $y \to \infty$. *Also* $\frac{y(e^y-y^2-1)}{2(e^y-1)} < \frac{1}{2}y$ *for all* $y > 0$.

Now we show that in the feedback system (3.44) *a trajectory starting in* $(x, y) \in Q$, *with* $y > 6$, *will always stay in* Q *(as long as it exists): To this end consider the horizontal distance between* $\varphi(\cdot, (x^0, y^0)^\top)$ *and the point* $\begin{pmatrix} \frac{1}{2}\varphi(t, (x^0, y^0)^\top)_2 \\ \varphi(t, (x^0, y^0)^\top)_2 \end{pmatrix}$ *on the half line* H. *This distance is just*

$$d(t) = \frac{1}{2}\varphi(t, (x^0, y^0)^\top)_2 - \varphi(t, (x^0, y^0)^\top)_1,$$

i.e., the difference between the x*-components of these points. By* (3.46) *the derivative of* $d(t)$ *satisfies*

$$\dot{d}(t) = \frac{1}{2}\tilde{g}(x, y, y) - f(x, y) > 0, \tag{3.47}$$

for $(x, y)^\top \in Q$. *Hence* $d(t)$ *is non decreasing on the set* Q. *By* (3.39) *and* (3.41) *we have* $\dot{x} > 0$ *and* $\dot{y} > 0$ *on* Q. *Moreover, the right hand side boundary of* Q *approaches* H, *i.e., the distance between this boundary and* H *decays monotonically for increasing values of* $y > 6$. *Hence the solution* $\varphi(\cdot, (x^0, y^0)^\top)$ *for* $(x^0, y^0)^\top \in Q$ *must stay in* Q, *as long as it exists. Now assume* $(x, y)^\top \in Q$ *with* $y \gg 1$ *sufficiently large and such that* $y > 2x + 1$. *By* (3.47) *if follows that* $\varphi_2(t) > 2\varphi_1(t) + 1$ *for all* $t \geq 0$ *along the trajectory* φ *with initial condition* (x, y). *Because of the dominating exponential term we have*

$$\begin{aligned} \nabla V(x, y)(f(x, y), \tilde{g}(x, y, y))^\top &= \frac{y^3(y - 2x)\exp(y^2)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)} \\ &> \frac{y^3 \exp(y^2)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)} > \frac{1}{4}y^4 = (V(x, y))^2 \end{aligned}$$

along trajectories, provided that y *is sufficiently large. Finally,* $\dot{V} > V^2$ *implies a finite escape time for the* y*-component. In Figure 3.3 the phase portraits of a few trajectories with finite escape time are plotted. Without the feedback interconnection the trajectories with same initial conditions would look as in Figure 3.4.*

Concluding, this example shows that although systems have the asymptotic gain property and the small gain condition is satisfied, Theorem 3.3.2 may not be applicable to the feedback interconnection, since solutions of the interconnected system do not exist for all times.

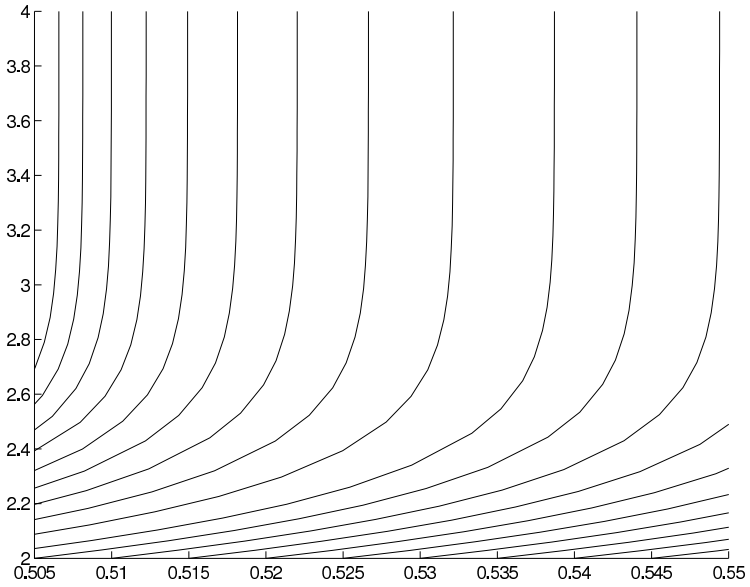


Figure 3.3.: Phase portrait of solutions with finite escape time of the feedback system in Example 3.3.9

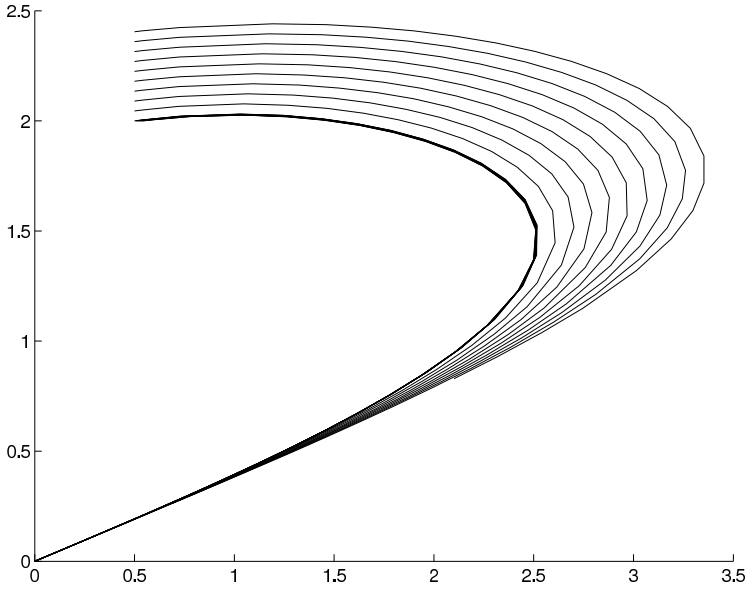


Figure 3.4.: Phase portrait of trajectories with the same initial values as in Figure 3.3, but this time of the original system by Hahn.

3.3.2. ISS for linear systems

The ISS small gain theorem of the preceding section simplifies to a linear matrix criterion, if the gains are all linear. From Lemma 2.0.1 we know that if $\mu = \Sigma$ and the gain matrix $\Gamma \in \mathcal{G}^{n\times n}$ is such that $\Gamma_\mu(s) = Gs$ for some $G \in \mathbb{R}_+^{n\times n}$, then the small gain condition (3.29) reduces to $\rho(G) < 1$, i.e., the spectral radius of the matrix G has to be less than one. This is because $\rho(G) < 1$ implies that for $\varepsilon > 0$ sufficiently small also $\rho(G \cdot (1+\varepsilon)I) < 1$, where $D(s) = (1+\varepsilon)s$ may serves as the diagonal operator in the small gain condition, so that we have $\Gamma_\Sigma \circ D(s) = G \cdot (1+\varepsilon)s$.

Analogously to Example 3.2.1 it is possible to estimate the ISS bounds for interconnected linear systems, hence we can take a closer look to what G looks like:

Consider the following setup, where in the sequel we omit the external input, formerly denoted by u, for notational simplicity. Let

$$\dot{x}_i = A_i x_i, \quad x_i \in \mathbb{R}^{N_i}, \quad i = 1, \dots, n,$$

describe n globally asymptotically stable linear systems, which are interconnected by the formula

$$\dot{x}_i = A_i x_i + \sum_{j=1}^{n} \Delta_{ij} x_j \quad i = 1, \dots, n,$$

using interconnection matrices $\Delta_{ij} \in \mathbb{R}^{N_i \times N_j}$, which can be rewritten as

$$\dot{x} = (A + \Delta)x, \tag{3.48}$$

where A is block diagonal, $A = \operatorname{diag}(A_i, i = 1, \dots, n)$, each A_i is Hurwitz and the matrix $\Delta = (\Delta_{ij})$ is also in block form and encodes the connections between the n subsystems. We suppose that $\Delta_{ii} = 0$ for all i. Define the matrix $R = (r_{ij}), R \in \mathbb{R}_+^{n\times n}$ by $r_{ij} := ||\Delta_{ij}||$, the induced matrix norm of Δ_{ij}. For each subsystem, there exist non-negative numbers M_i, λ_i such that $\|e^{A_i t}\| \le M_i e^{-\lambda_i t}$ for all $t \ge 0$. Define a matrix $D \in \mathbb{R}_+^{n\times n}$ by $D := \operatorname{diag}(\frac{M_i}{\lambda_i}, i = 1, \dots, n)$. From the last subsection we obtain the following implication of Theorem 3.3.4:

Corollary 3.3.10 *If $\rho(D \cdot R) < 1$ then the origin is globally asymptotically stable with respect to (3.48).*

Proof. Denote the initial value by x^0. Without loss of generality $t_0 = 0$. By elementary ODE theory we have

$$x_i(t) = e^{A_i t} x_i^0 + \sum_{j \ne i} \int_0^t e^{A_i (t-s)} \Delta_{ij} x_j(s) ds$$

and by standard estimates

$$\|x_i(t)\| \leq \|x_i^0\| M_i e^{-\lambda_i t} + \sum_{j \neq i} r_{ij} \frac{M_i}{\lambda_i} \|x_{j,[0,t]}\|. \tag{3.49}$$

As we see from (3.49), in this case the gain matrix happens to be $G = D \cdot R$. Now, as in the introduction of this Subsection, by Lemma 2.0.1 we have that there exists a $D = (1+\varepsilon)\mathrm{id}$ for some $\varepsilon > 0$, such that with $\Gamma_\Sigma(x) := Gx$ we have $\Gamma_\Sigma \circ D(x) \not\geq x$ for all $x \neq 0$ if and only if $\rho(G) < 1$. An application of Theorem 3.3.4 yields that (3.48) is ISS with respect to external inputs, i.e., in the absence of those, ISS reduces to global asymptotic stability of the origin. □

3.3.3. Local trajectory based estimates

Since in general it is probably difficult to check the small gain condition for large-scale networks globally, and also since arbitrary large values for x in the small gain condition might lack physical significance, it is desirable to have local versions of the preceding results.

In Chapter 4 we investigate a numerical method to check the small gain condition locally. Here we consider implications to the stability of interconnected systems, when the small gain condition holds only locally.

Given $w' \in \mathbb{R}^n_+$ we say that Γ_μ satisfies the *local small gain condition* on $[0, w']$, provided that

$$\Gamma_\mu(w') \ll w' \text{ and } \Gamma_\mu(s) \not\geq s, \ \forall\, s \in [0, w'],\ s \neq 0. \tag{3.50}$$

By Lemma 2.6.1 the local small gain condition holds, e.g., when we have a $w' \gg 0$ such that $\Gamma_\mu(w') \ll w'$ and $\Gamma_\mu^k(w') \to 0$ as $k \to \infty$.

Now we have all the ingredients we need to state the local version of the ISS small-gain theorem:

Theorem 3.3.11 *Assume that all subsystems in* (3.12)*, with $y_i = x_i$, for $i = 1, \ldots, n$, are locally ISS, i.e., for $i = 1, \ldots, n$ the estimate* (3.17) *holds locally:*

Assume that for $i = 1, \ldots, n$ there exists a constant $r_i > 0$ (possibly infinite), such that for all initial conditions $\|x_i(t_0)\| < r_i$ and (internal) inputs $\|x_j\|_\infty < r_i$, for $j \neq i$, the estimate

$$\begin{aligned} \|x_i(t)\| \leq \beta_i(\|x_i(t_0)\|, t - t_0) + \mu_i\big(\gamma_{i1}(\|x_{1[t_0,t]}\|_\infty), \ldots, \\ \gamma_{in}(\|x_{n[t_0,t]}\|_\infty)\big) + \gamma_i(\|u\|_\infty) \end{aligned} \tag{3.51}$$

holds for all $t \geq t_0$. Let Γ be given as before. Suppose Γ_μ satisfies (3.50).

Then the compound system is LISS, i.e., there exists a real number $r^* > 0$, functions $\hat{\beta} \in \mathcal{KL}$, and $\hat{\gamma} \in \mathcal{K}_\infty$, such that system (3.13) *satisfies*

$$\|x(t)\| \leq \hat{\beta}(\|x(t_0)\|, t - t_0) + \hat{\gamma}(\|u\|_\infty)$$

for all $t \geq t_0$, all $x(t_0) \in \mathbb{R}^N$ and all inputs $u \in L^\infty$ satisfying $\|x(t_0)\| < r^*, \|u\|_\infty < r^*$.

The proof comprises the following steps: At first we establish the existence of solutions for all times and find r^*. Next we establish that system (3.13) satisfies (3.7) and (3.5) for $\|x^0\|, \|u\|_\infty \leq r^*$, i.e., the compound system is locally stable and has the local asymptotic gain property. From here it follows that the corresponding unforced system is 0-AG, hence the composite system (3.13) is LISS.

Proof. Throughout the proof let $w^* \gg 0$ and φ be given by Theorem 2.6.3.

Step 0 – Forward completeness and $\mathbf{r}^*$ Define

$$\varepsilon := \sup\Big\{r \geq 0 : s \in \mathbb{R}^n_+, \|s\| < r \implies s \ll w^*\Big\}.$$

Let $\delta = \varphi^{-1}(\varepsilon/2)$. Let $\xi \in \mathbb{R}^{\sum_{i=1}^n N_i}$ be such that $\|\underline{\beta}(\|\xi\|, 0)\| < \delta/2$ and $\|\xi\| < \varepsilon/2$. Let $u \in L^\infty(\mathbb{R}_+; \mathbb{R}^M)$ as in (3.13) such that $\|\underline{\gamma}(\|u\|_\infty \cdot \underline{e})\| < \delta/2$.

Define $T^* := \inf\Big\{t \geq t_0 : \Big\|\Big\| x(\cdot, \xi, t_0, u)_{[t_0,t]} \Big\|\Big\| \geq \frac{1}{2}\varepsilon\Big\}$. Then the vector $\Big\| x(\cdot, \xi, t_0, u)_{[t_0,T^*]} \Big\|$ satisfies $\Big\|\Big\| x(\cdot, \xi, t_0, u)_{[t_0,T^*]} \Big\|\Big\| \leq \frac{1}{2}\varepsilon$, and therefore $\Big\| x(\cdot, \xi, t_0, u)_{[t_0,T^*]} \Big\| \leq w^*$ by the definition of ε. So we may apply Theorem 2.6.3 to the following inequality derived from (3.21), by taking the supremum over $t \in [t_0, T^*]$:

$$(\mathrm{id} - \Gamma_\mu)\left(\Big\| x(\cdot, \xi, t_0, u)_{[t_0,T^*]} \Big\|\right) \leq \underline{\beta}(\|\xi\|, 0) + \underline{\gamma}(\|u\|_\infty \cdot \underline{e}) \quad (3.52)$$

This leads to

$$\begin{aligned}\Big\|\Big\| x(\cdot, \xi, t_0, u)_{[0,T^*]} \Big\|\Big\| &\leq \varphi(\|\underline{\beta}(\|\xi\|, 0)\| + \|\underline{\gamma}(\|u\|_\infty \cdot \underline{e})\|) \\ &< \varphi(\delta/2 + \delta/2) = \varphi(\delta) \\ &= \varepsilon/2.\end{aligned}$$

This implies that there is no such minimal time T^*, such that the norm of the trajectory $x(\cdot, \xi, u)$ leaves the ε-neighborhood around the origin.

Hence this trajectory stays in that neighborhood for all times. This proves local forward completeness.

Now choose r^* such that $0 < r^* < \sup\{\|s\| : s \in \mathbb{R}^n, \|s\| < \varepsilon, \|\underline{\gamma}(s)\| < \delta/2, \|\underline{\beta}(s,0)\| < \delta/2\}$, then solutions for for initial conditions and external inputs in norm less than r^* exist for all times.

Step 1 – Establishing LS By construction r^* is such that for all $\|u\|_\infty < r^*$ and $\|\xi\| < r^*$ we have

$$\|x(\cdot,\xi,t_0,u)\|_\infty \leq \varphi(\|\underline{\beta}(\|\xi\|,0)\| + \|\underline{\gamma}(\|u\|_\infty \cdot \underline{e})\|)$$

which by the weak triangular inequality is

$$\leq \varphi(2\|\underline{\beta}(\|\xi\|,0)\|) + \varphi(2\|\underline{\gamma}(\|u\|_\infty \cdot \underline{e})\|)$$

and hence

$$\leq \sigma_1(\|\xi\|) + \sigma_2(\|u\|_\infty)$$

for some $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$. This establishes local stability.

Step 2 – Establishing LAG Using the GS estimate we can assure that

$$\|x(\cdot,\xi,t_0,u)\|_\infty < r^*,$$

which allows to use the LISS estimates repeatedly. To this end let $\sigma = \max\{\sigma_1,\sigma_2\} \in \mathcal{K}_\infty$, where σ_1 and σ_2 are given from step 1. Then for $u \in L^\infty$, $\|u\|_\infty < \min\{r^*, \sigma^{-1}(\frac{1}{3}r^*)\}$ and $\|\xi\| < \min\{r^*, \sigma^{-1}(\frac{1}{3}r^*)\}$ the GS estimate assures for any $t_1 \geq t_0$ that

$$\begin{aligned}\|x(t_1,\xi,t_0,u)\| &\leq \|x(\cdot,\xi,t_0,u)\|_\infty \\ &\leq 2\cdot\sigma\left(\min\left\{r^*, \sigma^{-1}\left(\frac{1}{3}r^*\right)\right\}\right) \\ &\leq \frac{2}{3}r^* < r^*.\end{aligned}$$

Hence the LISS estimate for the subsystems gives us for $t > t_1 > t_0$

$$\begin{aligned}\big|x(t,x(t_1,\xi,t_0,u),t_1,u)\big| \leq\ & \underline{\beta}\left(\big|x(t_1,\xi,t_0,u)\big|, t-t_1\right) \\ & + \Gamma_\mu\left(\Big\| x(\cdot,x(t_1,\xi,t_0,u),t_1,u)_{[t_1,t]}\Big\|\right) + \underline{\gamma}(\|u\|_\infty\cdot\underline{e}).\end{aligned}$$

For $t_1 = (t - t_0)/2$ this becomes

$$\begin{aligned}\left|x(t,\xi,t_0,u)\right| &= \left|x(t,x((t-t_0)/2,\xi,t_0,u),(t-t_0)/2,u)\right| \\ &\le \underline{\beta}\left(\left|x((t-t_0)/2,\xi,t_0,u)\right|,(t-t_0)/2\right) \\ &+ \Gamma_\mu\left(\left\| x(\cdot,x((t-t_0)/2,\xi,t_0,u),(t-t_0)/2,u)_{[(t-t_0)/2,t]}\right\|\right) \\ &+ \underline{\gamma}(\|u\|_\infty \cdot \underline{e})\end{aligned}$$

and taking the lim sup for $t \to \infty$, and by Lemma 3.3.8 we obtain

$$l(\xi,u) := \limsup_{t\to\infty}\left|x((t-t_0)/2,\xi,t_0,u)\right| \le \Gamma_\mu(l(\xi,u)) + \underline{\gamma}(\|u\|_\infty \cdot \underline{e}).$$

Since $l(\xi,u) \le w^*$ we again may apply Theorem 2.6.3 to get

$$\|l(\xi,u)\| \le \varphi(\|\underline{\gamma}(\|u\|_\infty \cdot \underline{e})\|) \le \sigma_2(\|u\|_\infty),$$

where σ_2 is taken from step 1, which establishes the local version of (3.6).

Step 3 – LISS In steps 1 and 2 we established that the origin of the unforced system

$$\dot{x} = f(x,0)$$

is stable and attractive, hence 0-AS. By Lemma 3.1.5 the composite system is LISS. □

3.3.4. IOpS Lyapunov functions

Now that we have derived trajectory based small gain theorems in the previous subsections, we now aim at a Lyapunov formulation of these small gain theorems.

Technically, this works as follows: We assume that a network of subsystems (3.12) is such that for each subsystem there exists an IOpS Lyapunov function satisfying (3.26) and (3.27), and the subsystem satisfies the unboundedness observability property (3.28). Our Lyapunov small gain theorem gives a (nonsmooth) IOpS Lyapunov function for the compound system in terms of the IOpS Lyapunov functions of the subsystems, provided that a small gain condition holds. To combine the Lyapunov functions of the subsystems to a Lyapunov function of the compound system we rescale each component Lyapunov function by a $\mathcal{K}_\infty$ function σ_i. The existence of these $\mathcal{K}_\infty$ functions is ensured by the results of Section 2.3: The small gain condition ensures the existence of a path in the decay set $\Omega(\Gamma_\mu)$, where Γ is the

gain matrix corresponding to the Lyapunov gains, similar as in the trajectory based scenario. Now this path can be chosen such that it can be parametrized by $\mathcal{K}_\infty$ functions σ_i. The Lyapunov function for the compound system then can be taken as the maximum of the scaled component Lyapunov functions,

$$V(x) = \max_i \sigma_i^{-1} \circ V_i(x_i).$$

Roughly speaking, the scaling ensures that if the maximum is attained for the ith subsystem then the decay condition (3.27) is satisfied for this particular subsystem, so that the differential $\nabla V(x) f(x,u)$ is essentially $\nabla V_i f_i(x,u) \leq -\alpha_i(V_i(x_i), \|x_i\|)$, so that V indeed is an IOpS Lyapunov function for the compound system.

We construct an IOpS Lyapunov function under the assumption, that each subsystem in the network has at least one input from within the network, or equivalently, that Γ has no zero rows.

Theorem 3.3.12 *Consider a network of IOpS systems as in* (3.12), (3.26), (3.27), *such that* $\Gamma \in \mathcal{G}^{n\times n}$ *given by* (3.27) *has no zero rows.*

Assume there exists a class $\mathcal{K}_\infty$*-function* η *such that for* $D = \operatorname{diag}_n(id+\eta)$ *we have*

$$D \circ \Gamma_\mu(s) \not\geqslant s, \quad \forall s \in \mathbb{R}^n_+, s \neq 0. \tag{3.53}$$

Then there exists an IOpS Lyapunov function[1] *for system* (3.13).

An immediate consequence is the following:

Corollary 3.3.13 *Consider network of IOS/ISpS/ISS systems as in* (3.12), (3.26), *and* (3.27), *i.e., with* $h_i = id$ *for all* i, *such that* $\Gamma \in \mathcal{G}^{n\times n}$ *has no zero rows. And let* $\mu \in \mathrm{MAF}_n^n$ *be given by* (3.27).

Assume there exists a class $\mathcal{K}_\infty$*-function* η *such that for* $D = \operatorname{diag}_n(id+\eta)$ *we have*

$$D \circ \Gamma_\mu(s) \not\geqslant s, \quad \forall s \in \mathbb{R}^n_+, s \neq 0. \tag{3.54}$$

Then there exists an IOS/ISpS/ISS Lyapunov function[1] *for system (3.13).*

Remark 3.3.14 *In* [JMW96] *a Lyapunov version of the ISpS small gain theorem is given. It states that if the small gain condition*

$$\exists c_0 \in \mathbb{R}_+ : \ \gamma_1 \circ \gamma_2(r) < r \text{ for all } r > c_0$$

is satisfied, then the feedback interconnection of two systems is also ISpS. Here $\gamma_1, \gamma_2 \in \mathcal{K}$ *are the Lyapunov gains between the systems. The Lyapunov*

[1] In fact, this Lyapunov function may not be Lipschitz near the origin, but still will be differentiable almost everywhere, see Remark 3.3.16.

function of the compound system is also given as a combination of the Lyapunov functions of the subsystems. The idea behind our onstruction follows that of [JMW96]*, but in a more general manner.*

In contrast to the results of [JMW96]*, our Lyapunov small gain theorem gives a Lyapunov function for the network for more than two interconnected subsystems and also covers the case of input-to-output practical stability.*

The condition

$$\Gamma_{\oplus}(s) \not\geq s \text{ for all } s \neq 0$$

with

$$\Gamma = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix}$$

is equivalent to the condition in [JMW96] *for* $c_0 = 0$*. Accordingly, the condition*

$$\exists D = \operatorname{diag}(id + \rho), \rho \in \mathcal{K}_\infty : \ D \circ \Gamma_\Sigma(s) \not\geq s \text{ for all } s \neq 0$$

again with

$$\Gamma = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix}$$

is equivalent to the condition of [JTP94] *with* $c_0 = 0$*, see Remark 3.3.6.*

The proof of Theorem 3.3.12 is divided into two steps. One is the construction of the functions σ_i, i.e., the scaling functions that were mentioned in the introduction, which was carried out in Section 2.3. The second step is put into the following proposition.

Proposition 3.3.15 *Consider a network of IOpS systems as in* (3.12)*,* (3.26)*, and* (3.27)*. For each subsystem* Σ_i*,* $i = 1, \ldots, n$*, let* V_i *be an IOpS Lyapunov function of the* i*th system satisfying (3.26) and (3.27).*

Assume there exists a diagonal operator $D = \operatorname{diag}_n(id + \alpha)$*,* $\alpha \in \mathcal{K}_\infty$*, and a locally Lipschitz path in* $\mathbb{R}^n_+$ *parameterized by* $\sigma \in \mathcal{K}^n_\infty$*, such that* σ_i^{-1} *is locally Lipschitz and* $(\sigma^{-1})'$ *fails to exist for at most countably many points, for* $i = 1, \ldots, n$*, and* σ *satisfies*

$$\begin{aligned} \sigma(t) \in \Omega(D \circ \Gamma_\mu) \quad & \text{for all } t > 0, \text{ and} \\ (\sigma_i^{-1})'(t) > 0 \quad & \text{for almost all } t > 0, \text{ for } i = 1, \ldots, n. \end{aligned} \tag{3.55}$$

Then for the composite system (3.13)

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}$$

is an IOpS Lyapunov function[2] *such that for arbitrary* $\kappa \in \mathcal{K}_\infty$ *we have that*

$$\begin{aligned} \alpha_1(\|h(x)\|) \le V(x) \le \alpha_2(\|x\|), \quad &\text{for all } x \in \mathbb{R}^N \\ &\text{and, for allmost all } x \in \mathbb{R}^N, \\ V(x) > \big(\rho^{-1} \circ (id + \kappa)\big)(\max_i \gamma_i(\|u\|)) + \big(\rho^{-1} \circ (id + \kappa^{-1})\big)(d) \\ \implies \nabla V(x) \cdot f(x,u) \le -\beta(V(x), \|x\|) \end{aligned} \tag{3.56}$$

for some positive $\mathcal{KL}$ *function* β *and functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, where the functions* γ_i *and* $d = \max_i d_i$ *are given by* (3.27) *and* $\rho \in \mathcal{K}_\infty$ *is defined in* (3.59) *in terms of* Γ_μ*,* α*, and* σ*.*

Remark 3.3.16 *The function* V *defined in Proposition 3.3.15 is not necessarily Lipschitz near the origin. This is not a problem, since the derivative of* V *along trajectories (which only exists almost everywhere) will be evaluated only away from the origin, anyway. On the other hand, there exist smoothing results for Lyapunov functions that are only continuous or not even that, see, e.g., the literature discussion given in the last section of* [LSW96]*. Unfortunately, at current it is not clear to us, whether or not such a smoothing is a priori possible, such that* V *is Lipschitz near the origin, so that the essential properties needed in the proof are maintained. However, we conjecture that it is indeed possible to mollify an IOpS Lyapunov function that is only Lipschitz everywhere except at the origin to a continuously differentiable IOpS Lyapunov function.*

Before we proceed with the proof of Proposition 3.3.15 we introduce a technical argument, which is an adaptation from [SW00, Lemma A.5].

Lemma 3.3.17 *Let* $\gamma \in \mathcal{K}_\infty$ *and* $V : \mathbb{R}^N \to \mathbb{R}_+$ *be a positive definite, locally Lipschitz continuous function that satisfies for almost all* $x \in \mathbb{R}^n$ *that*

$$V(x) \ge \gamma(\|u\|) \text{ and } V(x) \ne 0 \implies \nabla V(x) f(x,u) < 0, \tag{3.57}$$

with f *as in* (3.3)*. Then there exists a positive* $\mathcal{KL}$ *function* β *such that for almost all* $x \in \mathbb{R}^n$*,*

$$V(x) \ge \gamma(\|u\|) \implies \nabla V(x) f(x,u) \le -\beta(V(x), \|x\|).$$

Observe that in the following proof we will actually not need that V is Lipschitz near the origin (cf. the definition of Lipschitz near a point in the appendix, to avoid confusion). Instead all that is needed is that gradients

[2] Actually, V is not a Lyapunov function in the strict sense of our definition, as V may fail to be Lipschitz near the origin. However this does not produces problems, see Remark 3.3.16.

exist for allmost all points in a neighborhood of the origin, but they do not need to be bounded, as it would be the case if V were Lipschitz near 0.

Proof. First we need to remark that the function V is Lipschitz of rank K_C on every compact set $C \subset \mathbb{R}^N$ with K_C only depending on C, c.f. [Aul97, Satz 2.4.5].

Noting that the derivative of a locally Lipschitz continuous function exists almost everywhere (see Corollary A.6 in the Appendix), a particular consequence of the above observation is that ∇V exists almost everywhere and is essentially bounded on compact sets.

For $s, r \geq 0$ define the compact (possibly empty) sets

$$R(s,r) := \big\{(x,u) : \|x\| \leq r,\, V(x) \geq s,\, \gamma(\|u\|) \leq V(x)\big\}$$

The sets satisfy

$$\begin{aligned} s > s' &\implies R(s,r) \subset R(s',r) \\ r > r' &\implies R(s,r) \supset R(s,r') \end{aligned}$$

We now define

$$\beta_0(s,r) := \operatorname*{ess.\,min}_{(x,u)\in R(s,r)} -\nabla V(x) f(x,u)$$

with the convention that $\beta_0(s,r) = \infty$ if $R(s,r) = \emptyset$. The function β_0 is nonincreasing in r and nondecreasing in s and, for $s > 0$, $\beta_0(s,r)$ is positive by (3.57). Now define

$$\hat{\beta}(s,r) := \min\{\beta_0(s,r), s\}, \quad s, r \geq 0.$$

This function is finite valued, has the same monotonicity properties as β_0, and satisfies $\hat{\beta} \leq \beta_0$ as well as $\hat{\beta}(s,r) > 0$ if $s > 0$. Let $\hat{\beta}(s,r) = 0$ for $s < 0$ and define

$$\tilde{\beta}(s,r) := \int_{s-1}^{s} \hat{\beta}(\nu, r) d\nu, \quad s, r \geq 0.$$

Again this function has the same monotonicity properties, satisfies $\tilde{\beta}(s,r) > 0$ if $s > 0$ and is in addition continuous in s. But, it may not be strictly increasing in s nor need it decrease to zero as $t \to \infty$. We obtain a positive $\mathcal{KL}$ function β from $\tilde{\beta}$ by letting

$$\beta(s,r) := \frac{s}{1+s}\frac{1}{1+r}\tilde{\beta}(s,r), \quad s, r \geq 0.$$

The function β satisfies the desired properties by construction, since pointwise we have $\beta_0 \geq \hat{\beta} \geq \tilde{\beta} \geq \beta$ and

$$V(x) \geq \gamma(u) \implies \nabla V(x) f(x,u) \leq -\beta(V(x), \|x\|). \qquad \square$$

Now we can proceed with the proof of Proposition 3.3.15.

Proof of Proposition 3.3.15. By (3.55) we have

$$\sigma(t) \gg D \circ \Gamma_\mu(\sigma(t)), \quad \forall t > 0,$$

so that each function σ_i satisfies

$$\sigma_i(t) > (\mathrm{id} + \alpha)\Big((\pi_i \circ \Gamma_\mu)(\sigma(t))\Big), \tag{3.58}$$

for all $t > 0$ and $i = 1, \dots, n$. We define

$$M_i := \Big\{(v_1, \dots, v_n)^\top \in \mathbb{R}^n_+ : \sigma_i^{-1}(v_i) > \max_{j \neq i}\{\sigma_j^{-1}(v_j)\}\Big\}, \quad i = 1, \dots, n.$$

Notice that the sets M_i are open, and so are the sets $\left((V_1, \dots, V_n)^\top\right)^{-1}[M_i]$. From (3.58) it follows that

$$\sigma_i(t) - (\pi_i \circ \Gamma_\mu)(\sigma(t)) > \alpha\Big((\pi_i \circ \Gamma_\mu)(\sigma(t))\Big) =: \rho_i(t), \quad i = 1, \dots, n.$$

Note that each $\rho_i \in \mathcal{K}_\infty$ since Γ has no zero rows. Now let

$$\rho(t) = \min_i \rho_i(t), \tag{3.59}$$

which is again of class $\mathcal{K}_\infty$. Fix an arbitrary $\kappa \in \mathcal{K}_\infty$. By the weak triangular inequality for any $a, b > 0$ we have

$$\rho^{-1}(a + b) \leq \rho^{-1} \circ (\mathrm{id} + \kappa)(a) + \rho^{-1} \circ (\mathrm{id} + \kappa^{-1})(b). \tag{3.60}$$

Let $d = \max_i d_i$. The function V is locally Lipschitz by definition, but possibly not at the origin. So almost everywhere else the gradient ∇V is defined.

We are looking for a positive definite function $\tilde{\alpha}$, $\varphi \in \mathcal{K}_\infty$, and $\tilde{d} \in \mathbb{R}_+$ such that for almost all x, $V(x) > \varphi(\|u\|) + \tilde{d}$ implies $\nabla V(x) f(x, u) < -\tilde{\alpha}(V(x))$.

We claim that $\varphi(\|u\|) := \rho^{-1} \circ (\mathrm{id} + \kappa)(\max_i \gamma_i(\|u\|))$ and $\tilde{d} := \rho^{-1} \circ (\mathrm{id} + \kappa^{-1})(d)$ will do the job. To verify the defining implication of IOpS Lyapunov functions consider the inequality

$$\begin{aligned} V(x) &> \varphi(\|u\|) + \tilde{d} \\ &= \rho^{-1} \circ (\mathrm{id} + \kappa)(\gamma_i(|u|)) + \rho^{-1} \circ (\mathrm{id} + \kappa^{-1})(d) \end{aligned} \tag{3.61}$$

By (3.60) from this inequality it follows that $\rho(V(x)) > \gamma_i(|u|) + d$ or using the definition of ρ and d

$$\sigma_i(V(x)) - (\pi_i \circ \Gamma_\mu)(\sigma(V(x))) > \gamma_i(|u|) + d_i,$$

or equivalently, if $(V_1(x_1),\dots,V_n(x_n))^\top \in M_i$,

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &> (\pi_i\circ\Gamma_\mu)(\sigma(V(x))) + \gamma_i(|u|) + d_i \\ &= (\pi_i\circ\Gamma_\mu)\Big(\Big(\sigma_1\big(\sigma_i^{-1}(V_i(x_i))\big),\dots,\sigma_n\big(\sigma_i^{-1}(V_i(x_i))\big)\Big)^\top\Big) \\ &\quad + \gamma_i(|u|) + d_i \\ &> (\pi_i\circ\Gamma_\mu)\Big(\big(V_1(x_1),\dots,V_n(x_n)\big)^\top\Big) + \gamma_i(|u|) + d_i, \end{aligned} \tag{3.62}$$

where we have used $(V_1(x_1),\dots,V_n(x_n))^\top \in M_i$ in the last inequality.

Note that the sets $\overline{M_i}$ cover $\mathbb{R}^n$, hence $(V_1(x_1),\dots,V_n(x_n))^\top \in M_i$ is almost always true for some $i \in \{1,\dots,n\}$ (i.e., except on boundary points of the sets M_i, which are of Lebesgue measure zero in $\mathbb{R}^n$). Using continuity of V, σ, and Γ_μ, instead of (3.62) we then at least arrive at

$$V_i(x_i) \geq (\pi_i\circ\Gamma_\mu)\Big(\big(V_1(x_1),\dots,V_n(x_n)\big)^\top\Big) + \gamma_i(|u|) + d_i. \tag{3.63}$$

Now suppose that instead of Γ_μ we would apply all the above steps to $\tilde{\Gamma}_\mu := D_2\circ\Gamma_\mu$ and D replaced by D_1, where we obtain D_1, D_2 from D by Lemma 1.1.4, i.e., $D = D_1\circ D_2$. Then from (3.63) we would have

$$\begin{aligned} V_i(x_i) &\geq (\pi_i\circ\tilde{\Gamma}_\mu)\Big(\big(V_1(x_1),\dots,V_n(x_n)\big)^\top\Big) + \gamma_i(|u|) + d_i \\ &> (\pi_i\circ\Gamma_\mu)\Big(\big(V_1(x_1),\dots,V_n(x_n)\big)^\top\Big) + \gamma_i(|u|) + d_i, \end{aligned}$$

since Γ_μ has no zero rows. So without loss of generality, we may assume that (3.62) holds for some i.

Summarizing, this shows that (3.61) implies

$$V_i(x_i) > (\pi_i\circ\Gamma_\mu)\Big(\big(V_1(x_1),\dots,V_n(x_n)\big)^\top\Big) + \gamma_i(|u|) + d_i,$$

and hence from (3.27) we obtain by Corollary A.8 that

$$\begin{aligned} \nabla V(x) f(x,u) &= (\sigma_i^{-1})'(V_i(x_i))\nabla V_i(x_i) f_i(x,u) \\ &\leq -(\sigma_i^{-1})'(V_i(x_i))\alpha_i(V_i(x_i),\|x_i\|) < 0. \end{aligned}$$

for almost every x such that $V(x) \geq \varphi(\|u\|)$, so by Lemma 3.3.17 for almost every such x we even have

$$\nabla V(x) f(x,u) < -\beta(V(x),\|x\|)$$

for some $\mathcal{KL}$ function β.

The last step is to show that Corollary A.8 is indeed applicable, i.e., that $V_i^{-1}[U_{\sigma_i^{-1}}]$ has indeed zero Lebesgue measure: Note that the set $V_i^{-1}[r]$ has zero Lebesgue measure since by the Lyapunov decrease condition (3.27) every $V_i^{-1}[r]$, $r > 0$, is locally $(N_i - 1)$-dimensional. We know that the set $U_{\sigma_i^{-1}}$ where σ^{-1} fails to be differentiable is countable by assumption. Then $V_i^{-1}[U_{\sigma_i^{-1}}]$ is a countable union of null sets $V_i^{-1}[r]$, hence a null set. □

Proof of Theorem 3.3.12. We just combine the statements of Proposition 3.3.15 and Theorem 2.3.9: Recall that by Lemma 2.2.12 $D \circ \Gamma_\mu(s) \not\geq s$ for all $s \neq 0$ if and only if $\Gamma_\mu \circ D(s) \not\geq s$ for all $s \neq 0$. We may always decompose D into two diagonal operators D_1, D_2, see Lemma 1.1.4, such that $D_1 \circ D_2 = D$, where D_1, D_2 are also of the form $\mathrm{diag}_n(\mathrm{id} + \alpha_i)$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$.

So we have $D_1 \circ \Gamma_\mu \circ D_2(s) \not\geq s$ for all $s \neq 0$, which we write as $\tilde{\Gamma}_\mu \circ D_2(s) \not\geq s$ for all $s \neq 0$. Now apply Theorem 2.3.9 to obtain a $\mathcal{K}_\infty^n$-function σ, satisfying

$$D_1 \circ \Gamma_\mu(\sigma(t)) = \tilde{\Gamma}_\mu(\sigma(t)) < \sigma(t),$$

or, in other words, $\sigma(t) \in \Omega(D_1 \circ \Gamma_\mu)$, for all $t > 0$.

In particular, this function σ is locally Lipschitz and on every compact interval in $(0, \infty)$ it is piecewise linear. Hence the inverse functions σ_i^{-1} piecewise linear on compact sets in $(0, \infty)$ and so they are locally Lipschitz on $(0, \infty)$ and satisfy $(\sigma_i^{-1})' > 0$ almost everywhere.

It remains to apply Proposition 3.3.15: The prerequisites of the theorem together with the function σ just discovered, we obtain an IOpS Lyapunov function V from Proposition 3.3.15 for the network, i.e., the network is also IOpS. □

3.4. Notes and references

For an exposition on the basic properties of input-to-state stability, equivalent and related concepts, as well as an extensive bibliography we recommend [Son06] for further reading.

The definition of IOpS and IOS Lyapunov functions are taken from [JTP94, SW00]. Much of the material in Section 3.3 has already been published in [DRW07]. The first small gain theorems are due to [JTP94] for the trajectory based IOpS version and [JMW96] for the ISpS Lyapunov (max) version. A notion of stability equivalent to ISS is input-to-state dynamical stability (ISDS). For the ISDS formulation a small-gain theorem yielding GAS of the interconnection is given in [Grü02b]. Other ISS small gain theorems are, e.g.,

[CPT95], where feedback loops of GS systems are shown to be GAS under a small gain condition.

The papers [Grü02b, JTP94, JMW96] use different approaches to the formulation of small gain conditions that yield sufficient stability criteria: In [JTP94] the proof is trajectory-based and incorporates sums of $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions. This approach requires that the composition of the gains is smaller than the identity in a robust sense, i.e., a for an arbitrary "small" class $\mathcal{K}_\infty$ function ρ the cycle $\gamma_{12} \circ \gamma_{21}$ has to be less than the term $(\mathrm{id} + \rho)^{-1}$. We have shown in Example 3.3.7 that within the context of the summation formulation of ISS and also more general formulations the robustness condition cannot be weakened. The ISS result of [JTP94] has turned out to be a special case of our results. For the maximum formulation of ISS as in [JMW96] we found that the cycle condition or, respectively, $\Gamma_\oplus(s) \not\geq s$ for all $s \neq 0$, is indeed sufficient. Moreover, whenever we have that solutions of monotone inequalities as in Theorem 2.5.4 are bounded by $\mathcal{K}_\infty$ functions, then our condition, $\Gamma_\oplus(x) \not\geq x$ for all $x \neq 0$, is also necessary. So some sense, the small gain condition is quite natural, even in higher dimensions.

General stability conditions for large scale interconnected systems have been obtained by various authors in other contexts. In [MM77] sufficient conditions for the asymptotic stability of a composite system are stated in terms of the negative definiteness of some test matrix. This matrix is defined through the given Lyapunov functions of the interconnected subsystems. Similarly, in [RHL77] conditions for the stability of interconnected systems in terms of Lyapunov functions of the individual systems are obtained.

In [Šil79] Šiljak considers structural perturbations and their effects on the stability of composite systems using Lyapunov theory. The method is to reduce each subsystem to a one-dimensional one, such that the stability properties of the reduced representation imply the same stability properties of the original interconnected system. In some cases the lower dimensional representation gives rise to an interconnection matrix $\bar{W}$, such that quasi dominance or negative definiteness of $\bar{W}$ yield asymptotic stability of the composite system.

In [Vid81] small gain type theorems for general interconnected systems with linear gains can be found. These results are of the form that the spectral radius of a gain matrix should be less than one to conclude stability. The result obtained here may be regarded as a nonlinear generalization in the same spirit. On the other hand, Corollary 3.3.10 is a special case of a theorem, which can be found in [Vid81, p. 110]. There the condition on the spectral radius is quite the same, although it is applied to a *test matrix*, whose entries are finite gains of products of *interconnection operators* and corresponding *subsystem operators*. These gains are non-negative numbers and, roughly

speaking, defined as the minimal possible slope of affine bounds on $\|G(x)\|_{L_\infty}$ with respect to $\|x\|_{L_\infty}$ for operators $G : L_\infty \to L_\infty$ and trajectories $x \in L_\infty$.

It is noteworthy, that Corollary 3.3.10 is also a consequence of [KHP06], where μ-analysis is used to investigate block perturbations of linear systems. Namely, the interconnection structure of the individual systems may be seen as a block perturbation of the system $\dot{x} = Ax$, and the condition $\rho(D \cdot R) < 1$ suffices for the perturbation Δ to stay within the open ball determined by the stability radius of A.

In the discrete time context Teel [Tee05] proves that if we have maximum ISS estimates of the type (3.17) (with $\mu = \oplus$ and $y_i = x_i$, $d_i = 0$) for each subsystem and if each cycle (or equivalently, each minimal cycle) in Γ is a contraction then the network under consideration is input-to-state stable. This result extends in a straightforward manner to continuous time systems.

For the maximum formulation of ISS in [PAQ03] Potrykus et al. prove a similar but rather involved small gain theorem for continuous time systems; their small gain condition is not at first sight equivalent to the one in [Tee05], but in [Pot02] Potrykus proves that the cycle condition is sufficient for the input-to-state stability of the interconnection.

There are stability concepts that are strongly related to ISS: Integral-input to state stability (iISS) and integral-input to integral-state stability. While the latter is in fact equivalent to ISS, the first one is strictly weaker, see [Son98]. It is known, that even cascades of iISS systems do not need to be iISS. Nonetheless, for certain interconnection types small gain type theorems are available for iISS systems, see [Ito06].

The construction of an IOpS Lyapunov function as in Proposition 3.3.15 and also the construction of a function σ that somehow separates the state space into regions, where the Lyapunov decrease conditions of the subsystems apply, is strongly motivated and follows to some extent the work by Jiang, Mareels, and Wang [JMW96], while extending their original result for two feedback interconnected subsystems to arbitrarily many systems, cf. Remark 3.3.14. The book by Šiljak [Šil79], and in particular Chapter 2 therein, also provides Lyapunov type stability theorems for large-scale systems, but concentrates more on connective stability with respect to structural disturbances rather than on notions on stability of sets depending on external inputs or disturbances to single subsystems.

4. Implementation and Application

In this chapter we want to analyze input-to-state stability and related stability properties of large-scale interconnected systems numerically. Let a large-scale interconnected systems be given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, y_2 \ldots, y_n, u), & y_1 &= h_1(x_1), \\ &\vdots & &\vdots \\ \dot{x}_n &= f_n(y_1, \ldots, y_{n-1}, x_n, u), & y_n &= h_n(x_n), \end{aligned} \tag{4.1}$$

which can be seen as one composite system

$$\dot{x} = f(x, y, u), \quad y = h(x). \tag{4.2}$$

We assume that each system in (4.1) is input-to-state stable or stable in a related sense (e.g., IOpS or ISS), e.g., as in (3.17). We also assume that a gain matrix $\Gamma \in \mathcal{G}^{n\times n}$ and a monotone aggregation map $\mu \in \mathrm{MAF}_n^n$ are given (again by, e.g., (3.17)). We verify the small gain condition locally, based on the operator Γ_μ that is provided, and calculate ISS gains from external inputs to the composite state of the network. If in addition Lyapunov functions for the individual systems are provided, we also consider numerical methods to implement Lyapunov functions for such networks. This allows to utilize the results of Chapters 2 and 3 using a computer.

It is known how to calculate external gains in the case of pure cascades, see e.g., [TKP02a, TKP02b]. In this case no small-gain condition has to be checked. If the internal gain functions, i.e., the gain functions γ_{ij} assembling $\Gamma \in \mathcal{G}^{n\times n}$, are all linear and $\mu = \Sigma$ or $\mu = \oplus$, then Γ_μ is linear or max-linear. Then we can write $\Gamma_\Sigma(x) = Gx$ or, respectively, $\Gamma_\oplus(x) = (\max_j g_{1j}x_j, \ldots, \max_j g_{nj}x_j)^\top$, for a nonnegative matrix $G \in \mathbb{R}_+^{n\times n}$. In this case testing the small gain condition reduces to verify that the spectral radius or, respectively, the maximum circuit geometric mean[1] of G is less than one. Numerically more efficient is the power method: For $\varepsilon > 0$ sufficiently small the matrix $\tilde{G} = G + \varepsilon E$, where E is the matrix consisting entirely of ones, has spectral radius or maximum circuit geometric mean less than one, if G does.

[1] See [Lur05] for the general definition.

Then there exists a vector $v^* \in \mathbb{R}^n_+$, $v^* \gg 0$, such that for any $v \in \mathbb{R}^n_+, v > 0$, the sequence

$$\left\{ \frac{1}{\|\tilde{G}^k v\|} \tilde{G}^k v \right\}_{k=0,1,2,\ldots}$$

converges to v^* as $k \to \infty$. The vector v^* is the unique positive eigenvector of $\tilde{G}$ to the simple eigenvalue $\lambda = \rho(\tilde{G})$. It is easy to see that $\Gamma_\Sigma(t \cdot v^*) \ll \tilde{G}t \cdot v^* \ll t \cdot v^*$ for any $t > 0$, i.e., the map $t \mapsto t \cdot v^*$ is a parameterized path with image in $\Omega(\Gamma_\Sigma)$. For $\Gamma_\oplus$ this works analogously. In the nonlinear case there exists algorithms that implement the Knaster-Kuratowski-Mazurkiewicz Theorem. Such an algorithm yields a point in the decay set $\Omega(\Gamma_\mu)$, and from here we can apply the theory developed in Chapter 2.

We propose a step by step approach to test input-to-state stability of the origin with respect to system (4.2). Based on Lemma 2.6.1 we can check the condition

$$\Gamma(s) \not\geq s, \text{ for } s \neq 0,$$

locally, i.e., for $s \in [0, w]$ for arbitrary $w \in \mathbb{R}^n_+$. Once we know that the small gain condition (for now only in this weak form without the operator D) holds, we know that the KKM Theorem, or more specifically, Proposition 1.5.3 is applicable, i.e., there exists a point in $\Omega(\Gamma_\mu)$. Now we implement an algorithm proposed by Eaves [Eav72] to find such a point $w' \leq w$. By Theorem 2.6.2 we now construct an operator D such that

$$\Gamma_\mu(D(s)) \not\geq s$$

for $s \in [0, w'], s \neq 0$. This makes the local ISS small-gain Theorem 3.3.11 applicable, leading to stability estimates for the composite system.

In this chapter we provide a theorem stating that convergence of

$$\Gamma^k_\mu(w) \to 0$$

as $k \to \infty$ implies the desired stability property of the composite system (4.2) locally. We provide pseudo code to demonstrate how the results of Chapters 2 and 3 can be implemented on a computer. Further we provide numerical examples to demonstrate the feasibility of our proposed methods.

4.1. Computational Methods

In this section we provide pseudo code for the computational steps described in the introduction that are related to verifying the small gain condition,

finding points, and constructing diagonal operators and paths in Ω. In the next section we state theorems to show how these steps can be used to verify stability properties of ISS networks. Throughout this section we assume that a matrix $\Gamma \in \mathcal{G}^{n\times n}$ and $\mu \in \mathrm{MAF}_n^n$ are given. In Subsection 4.1.2 we need that Γ has a zero-diagonal. Recall the definition of the decay sets

$$\Omega_i = \{x \in \mathbb{R}_+^n : \Gamma_\mu(x)_i < x_i\} \tag{4.3}$$
$$\Omega = \{x \in \mathbb{R}_+^n : \Gamma_\mu(x) \ll x\} \tag{4.4}$$
$$\Psi_i = \{x \in \mathbb{R}_+^n : \Gamma_\mu(x)_i \leq x_i\} \tag{4.5}$$
$$\Psi = \{x \in \mathbb{R}_+^n : \Gamma_\mu(x) \leq x\} \tag{4.6}$$

We make a notational convention for pseudo code: The symbol $x \leftarrow y$ denotes that the variable x is assigned the value of the variable y.

4.1.1. Testing the no joint increase condition locally

It all begins with the verification of the no joint increase condition. By assumption we already have that $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is a monotone operator satisfying $\Gamma_\mu(0) = 0$, see Lemma 1.2.3. We want to check the no joint increase condition locally, i.e., that Γ_μ satisfies

$$\Gamma_\mu(x) \not\geq x \tag{4.7}$$

for all $x \neq 0$, $x \in [0, w]$, where w is supposed to be sufficiently large. Following Lemma 2.6.1 Algorithm 1 can be used to establish exactly that.

The idea is to consider the sequence $\{\Gamma_\mu^k(w)\}_{k\in\mathbb{N}}$ and check that it converges to zero. In a numerical algorithm this corresponds to testing whether or not $\Gamma_\mu^k(w)$ can be made smaller than a small constant, that is chosen in advance with respect to the machine precision, by choosing k sufficiently large.

This algorithm yields only a sufficient condition for the no joint increase condition. Provided that it is successful, by Proposition 1.5.3 we can expect to find a point $w' \in \Omega \cap S_r$ for $r > 0$ given by $r = \min_i w_i$. On the other hand, if the algorithm is unsuccessful, then we do not have to expect that procedure in the next section works.

4.1.2. Finding a point in $\Omega \cap \mathrm{S_r}$

Given that Γ_μ satisfies the no joint increase condition (4.7) on the set $[0, w]$ for some $w \gg 0$ we know that by Proposition 1.5.3 there exists a point $w' \in \Omega \cap S_r$ for $r > 0$ given by $r = \min_i w_i$. The question is, how we can find this point

Algorithm 1 Verify the no joint increase condition (4.7) locally.

Require: $\Gamma \in \mathcal{G}^{n\times n}, \mu \in \mathrm{MAF}_n^n$
Require: $\varepsilon > 0$ sufficiently small /* to check if $\Gamma_\mu^k(w)$ tends to zero */
Require: MAXsteps $\in \mathbb{N}$ sufficiently large /* maximal number of steps allowed, we do not want to run the repeat loop forever */
pick $w \in \mathbb{R}_+^n, w \gg 0$ arbitrarily large
$k \leftarrow 0$
$x \leftarrow w$
repeat
$k \leftarrow k + 1$
$x \leftarrow \Gamma_\mu(x)$
until $\|x\| < \varepsilon$ or $k >$ MAXsteps
if $\|x\| < \varepsilon$ **then**
Γ_μ satisfies the condition (4.7) on $[0, w]$ /* successful */
else
restart the algorithm with a different choice of w /* algorithm failed */
end if

if Γ_μ is not linear or max-linear? In the nonlinear case we have to rely on algorithmic versions of the KKM Theorem 1.5.7, a topological result about interconnections of sets on simplices, which we used for Proposition 1.5.3 to find a point in $\Omega \cap S_r$. As the proof of the KKM Theorem provides merely an existence argument rather than a computational reasonable tool to find a point in the desired intersection, we rely on a more tractable and efficient algorithm proposed by Eaves in [Eav72]. For notational simplicity we write $S = S_r$. From now on $\{e^i\}_{i=1}^n$ denotes the standard basis of $\mathbb{R}^n$.

Roughly speaking, the argument in the proof of the KKM Theorem is as follows: Given an arbitrary small $\varepsilon > 0$ the simplex is refined by a grid of smaller simplices of diameter less than ε. For any ε it is possible to show that there exists a *special* simplex in this refinement of diameter less than ε. We will explain what we mean by *special* in a moment. Now as ε tends to zero, one obtains a point that is special, in our case, a point that is in the intersection of the sets Ω_i. But it turns out, that this procedure does not scale nicely: Neither is it possible to reuse the information for the refinement for $\varepsilon > 0$ when S is going to be refined for $\delta > 0$ with $\delta < \varepsilon$. Nor is it possible to apply this procedure iteratively, i.e., to refine every obtained *special* simplex as if it was S itself, as is also indicated in Figures 4.2 and 4.3: A simplex is *special*, if its vertices can be numbered from 1 to n such that vertex lies

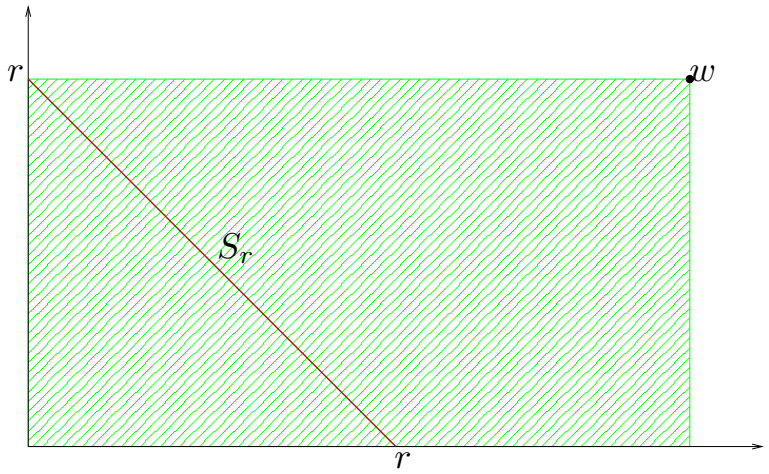


Figure 4.1.: The region $[0, w] \subset \mathbb{R}^n_+$ where the no joint increase condition can be verified with Algorithm 1.

in the set Ω_i. The simplex $S = \mathrm{co}\{r \cdot e_i\}_{i=1}^n$ is special, because each vertex $r \cdot e_i$ lies in Ω_i (this is a consequence of the zero diagonal of Γ). By iterative refinement one may end up with a simplex as in Figure 4.3. The simplex in this picture is special, but it does not cover the intersection set of interest. That means, that further refinements of this simplex cannot not lead to a point in this intersection set.

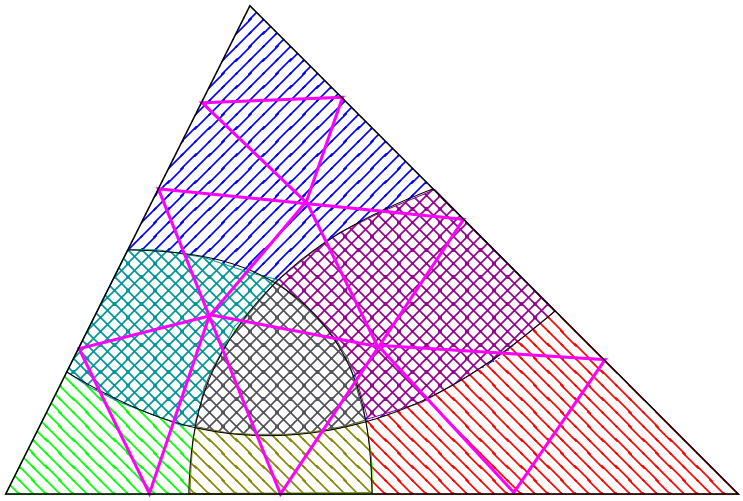


Figure 4.2.: In the proof of the KKM Theorem in Section 1.5 a simplex is triangulated by smaller simplices of diameter less than $\varepsilon > 0$. For each ε there exists at least one special simplex in the refinement.

Fortunately, there exist constructive algorithms to find fixed points of point

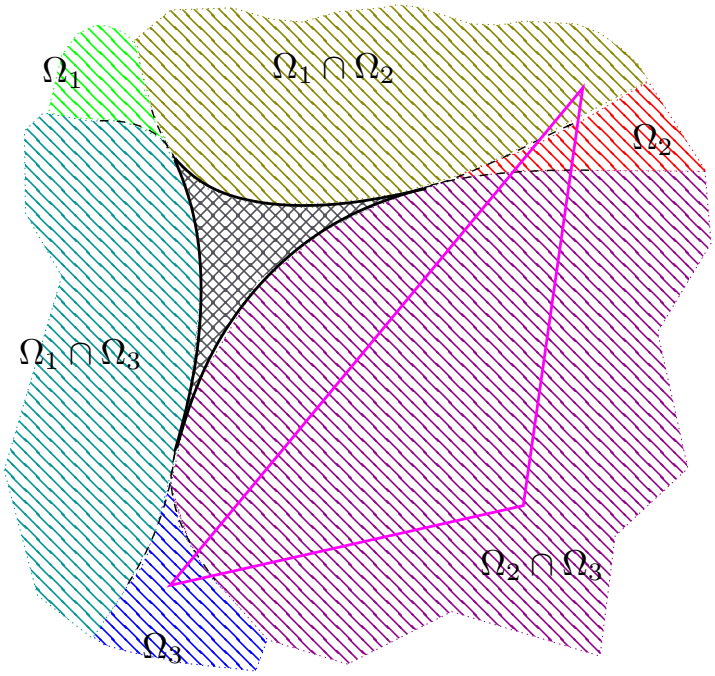


Figure 4.3.: A bad choice of *iterative* refinement steps may result in a *special* simplex that does not overlap with the intersection set of interest: Although the vertices of the pink simplex can be numbered according to the sets Ω_i such that each $i = 1, 2, 3$ appears exactly once (i.e., clockwise starting with $i = 1$ for the top vertex), the intersection $\Omega_1 \cap \Omega_2 \cap \Omega_3$ in the middle of the picture is clearly not contained in the simplex.

to set functions $f : S \to S^*$, where S is a simplex and S^* is the set of all subsets of S. A fixed point of such a function is a point $x \in S$ satisfying $x \in f(x)$. And it is known, that the KKM Theorem is equivalent to certain fixed point theorems, see e.g. [Eav72, Yan99]. In the next two subsections we provide an adaptation of Eaves algorithm [Eav72] to our setting. We have to introduce quite a bit of notation. We start with the definition of l-complete sets and proper labelings and provide a function $f : S \to S^*$ whose fixed points are exactly those points in $\Psi \cap S$.

Notation Recall that S is a $(n-1)$-simplex (i.e., a $(n-1)$-dimensional space). For notational simplicity we assume that S is the standard simplex S_1, i.e., $r = 1$. The pseudo code provided in Section 4.1.2 will work for general $r > 0$.

A *face* σ of S is a (possibly lower dimensional) simplex contained in S such

that $\sigma = \mathrm{co}\{e^i : i \in I\}$, where I is a subset of $\{1, \ldots, n\}$. A *facet* of S is one of the n faces of dimension $(n-2)$ of S, and S_i is the facet opposite of e^i.

It is important to note that the vector of barycentric coordinates of a point in S is hence the point itself in standard coordinates.

Let $S - S = \{s \in \mathbb{R}^n : s = s^1 - s^2,\ s^1, s^2 \in S\}$ and let $l : S \to (S-S)^*$ be a point to set map, carrying points in S to subsets of $S - S$. A subset C of S is called *l-complete* (or just *complete* if the reference to l is clear from the context) if there are vectors $l_i \in \bigcup_{y \in C} l(y)$ for $i = 1, \ldots, k$, for some k, such that the system

$$\sum_{i=1}^{k} l_i \lambda_i = 0, \quad \sum_{i=1}^{k} \lambda_i = 1$$

has a solution $0 \leq (\lambda_1, \ldots, \lambda_k)^\top \in \mathbb{R}^k$.

The function l defines a *proper labeling* of the function $f : S \to S^*$ if

1. the set of vertices of S is complete;
2. no facet of S contains a complete set;
3. if a sequence of complete sets tends to $x \in S$, then x is a fixed point of f.

If $\{x\}$ is a complete set, then x is called a *complete point*.

We quote the following result from [Eav72]:

Theorem 4.1.1 *If $f : S \to S^*$ has a proper labeling then it has a fixed point.*

We also quote an Example from [Eav72]:

Example 4.1.2 (KKM covering theorem) *For $i = 1, \ldots, n$ define l_i to be the ith column of the $n \times n$-matrix*

$$\begin{pmatrix} +1 & -1 & 0 & & 0 \\ 0 & +1 & -1 & & \vdots \\ \vdots & 0 & +1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & & -1 \\ -1 & 0 & 0 & & +1 \end{pmatrix}.$$

Let $C_1, \ldots, C_n$ be a closed covering of S such that $S_i \subset C_i$ for $i = 1, \ldots, n$. Let $f(x) = co\{e_i : x \in C_i\}$.

Let $C_{n+1} = C_1$ and define $l(x) = l_{i(x)}$, where

$$i(x) = \min\left\{j \in \{1, \ldots, n\} : x_j > 0 \text{ and } x \in C_{j+1}\right\}$$

Then l is a proper labeling of f, hence f has a fixed point, or equivalently, the intersection $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

We utilize the idea of the Example 4.1.2 to find a point in $\Psi \cap S$. We will rely on the following observation:

Remark 4.1.3 *Using the same vectors l_i, $i = 1\ldots, n$, as in Example 4.1.2 it holds that $\sum_{i=2}^{n} l_i = -l_1$. Consequently, a set M is complete for a proper labeling l that only takes values $l_1, \ldots, l_n$ if and only if there exist $x_1, \ldots, x_n \in M$ such that $l(x_i) = l_i$ for $i = 1, \ldots, n$.*

Keeping the notation, we obtain a proper labeling. Recall that $\Psi_j = \{x \in \mathbb{R}^n_+ : \Gamma_\mu(x)_j < x_j\}$ for $j = 1, \ldots, n$.

Lemma 4.1.4 *The function given by*

$$l(x) := l_{i(x)} \tag{4.8}$$

with

$$i(x) := \min\{j \in \{1, \ldots, n\} : x_j > 0, x \in \Psi_j\} \tag{4.9}$$

is a proper labeling for S.

Proof. We have to check that the set of vertices of S is complete with respect to l, that no facet of S is complete, and that if a sequence of complete sets $M^k \subset S$ decreases to a point $x \in S$ then

$$x \in \Psi = \bigcap_{j=1}^{n} \Psi_j.$$

For $j = 1, \ldots, n$ we have $i(e_j) = j$, as the jth component of e_j is the only positive entry and clearly $\Gamma_\mu(e_j)_j = 0 < (e_j)_j$, due to the assumption that Γ has a zero diagonal. Hence the set of vertices of S is complete.

Let M be a facet of S. Then there exists a $j \in \{1, \ldots, n\}$ so that $x_j = 0$ for all $x \in M$. Hence for no $x \in M$ we can have $i(x) = j$. By Remark 4.1.3 the set M cannot be complete.

Assume that for a sequence of sets $M^k \subset S$, $k \in \mathbb{N}$, we have $M^k \to x \in S$ and that every set M^k is complete. By Remark 4.1.3 for every $k \in \mathbb{N}$ there exist points $x_1^k, \ldots, x_n^k \in M^k$ such that $i(x_j^k) = j$, for each $j = 1, \ldots, n$. By assumption for every $j = 1, \ldots, n$ we have $\lim_{k\to\infty} x_j^k = x$. The sets Ψ_j are closed, so it follows that $x \in \Psi_j$. Together this gives $x \in \Psi = \bigcap_{j=1}^{n} \Psi_j$.

This concludes the proof. □

Note also that for actual computations it is sufficient to consider $l(x) = i(x)$, where $i(x)$ has been defined in (4.9). This follows from Remark 4.1.3,

i.e., instead of checking that a sum of labels is zero, we may instead keep a list of label indices and check, whether all possible indices are in that list. This reduces the need of memory needed to store labels by factor n.

Eaves fixed point algorithm

Now we explain the fixed point algorithm by Eaves, adapted to the function $f : S \to S^*$ defined in Example 4.1.2. In the next section we provide pseudo code that can be implemented on a computer.

Notation Let K^0 be a set of points called vertices. A *simplicial complex* K, or complex for short, with vertices K^0 is a collection of finite subsets of K^0, such that

1. $v \in K^0$ implies $\{v\} \in K$;
2. $\emptyset \neq \tau \subset \sigma \in K$ implies $\tau \in K$;
3. $\{\sigma \in K : v \in \sigma\}$ is finite for any $v \in K^0$.

A set $\sigma \in K$ with $k+1$ vertices is a k-simplex. Vertices are equal to 0-simplices. If $\tau \subset \sigma \in K$ then τ is a *face* of σ. Two distinct simplices of K are *adjacent* in K if they are both faces of a common simplex. The complex K is an n-dimensional *pseudomanifold* if

1. for all $\tau \in K$ there is an n-simplex σ such that $\tau \subset \sigma$;
2. if $\tau \in K$ is an $(n-1)$-simplex then τ is a face of exactly one or two n-simplices.

From now on we only consider n-dimensional pseudomanifolds.

Those $(n-1)$-simplices in only one n-simplex are termed *boundary* and the others *interior* $(n-1)$-simplices. Let $|K|$ denote the set of functions $\alpha : K^0 \to [0,1]$ such that

1. $\{v \in K^0 : \alpha(v) > 0\} \in K$;
2. $\sum_{v \in K^0} \alpha(v) = 1$.

If $\sigma \in K$ then $|\sigma| := \{\alpha \in |K| : \alpha(v) > 0 \implies v \in \sigma\}$. The terminology of K carries over to $|K|$ and the set $|K|$ is metrizable by

$$d(\alpha, \beta) = \left(\sum_{v \in K^0} (\alpha(v) - \beta(v))^2 \right)^{1/2}.$$

Let $L : K^0 \to (S - S)$. Then $\sigma \in K$ is termed *very complete* if there exists an $\varepsilon_\sigma > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_\sigma$ there exists an $\alpha \in |\sigma|$ such that

$$\sum_{v \in K^0} L(v)\alpha(v) = \begin{pmatrix} -\varepsilon - \varepsilon^2 - \ldots - \varepsilon^n \\ \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^{n-1} \end{pmatrix}.$$

It is possible to show that only $(n-1)$ and n-simplices of K can be very complete.

To understand the connection between proper labelings and very complete simplices, we kindly ask the reader to wait until Remark 4.1.6.

Eaves proves the following theorem (cf. [Eav72, p.7]):

Theorem 4.1.5 *Let K and $L : K^0 \to S - S$ be given. If σ_0 is the only very complete boundary $(n-1)$-simplex of K, then there exists a unique sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ of maximal length of distinct very complete adjacent $(n-1)$-simplices in K. Furthermore this sequence is infinite.*

This result is the theoretical basis of the algorithm: All the algorithm does is to find this particular sequence of very complete adjacent $(n-1)$-simplices in K. The simplices σ_l can be projected back to S and there their diameter shrinks down to zero, so that ultimately these sets converge to a fixed point of f.

More rigorously, we define a *triangulation* of $S \times [1, \infty)$ to be a homeomorphism $h = (h_1, h_2) : |K| \to S \times [1, \infty)$, where K is a complex. A *facet* of $S \times [1, \infty)$ is one of the $(n+1)$ faces of dimension $(n-1)$ and the boundary of $S \times [1, \infty)$ is the union of its facets. For $\sigma \in K$ denote by $h(\sigma)$ the set $\{h(|v|) : v \in \sigma\}$, and similarly for h_1, h_2.

The algorithm

We still have not fixed a particular pseudomanifold K yet and further postpone that another subsubsection.

Let $l : S \to (S - S)^*$ be a proper labeling and $h = (h_1, h_2) : |K| \to S \times [1, \infty)$ be a triangulation such that

1. $h^{-1}(S \times 1)$ is a simplex in $|K|$, say $|\sigma_0|$;
2. each boundary simplex of K is carried by h into some facet of $S \times [1, \infty)$;
3. if $h_2(\sigma_k)$ tends to infinity for a sequence $\sigma_k \in K$ then the diameter of $h_1(\sigma_k)$ tends to zero.

Now define $L : K^0 \to S - S$ by $L(v) \in l(h_1(|v|))$. The initial data will be described together with the complex K_1.

Roughly speaking, the algorithm now consists of stepping from one σ_l to the next σ_{l+1} such that $h_2(\sigma_l)$ tends to infinity. To this end let $\tau = \{v^0, \ldots, v^n, v^{n+1}\}$ be the n-simplex that contains $\sigma_l = \{v^0, \ldots, v^n\}$, but not σ_{l-1} $(l > 0)$. In this particular application of that algorithm, we find that if $L(v^{n+1}) = l_i$ then the vertex of σ_l labeled l_i has to be dropped. Hence we have to find a "new" v^{n+1} to form a new n-simplex that contains σ_l and another $(n-1)$-simplex, which will be σ_{l+1}, and so on.

Remark 4.1.6 *A priori there is no connection between proper labelings and very complete simplices. Both are defined independently of each other. The connection between the two concepts has not been made until this subsubsection.*

Here we defined $L : K^0 \to S - S$ by $L(v) \in l(h_1(|v|))$, where l is a proper labeling and h_1 is given by the triangulation. This assignment implies that σ_0 is a very complete simplex, where σ_0 is the initial simplex defined above in step 1. Theorem 4.1.5 now gives a sequence of complete simplices $h_1(\sigma_k)$ in S, and, since $h_2(\sigma_k) \to \infty$ (which can be shown, again using Theorem 4.1.5), this sequence decreases to a point $x \in S$. And in our particular application, this point x lies in the intersection $\bigcap_{j=1}^{n} \Psi_j$.

The complex $\mathbf{K_1}$ Let $D = \{x \in \mathbb{R}^n_+ : \|x\|_1 \geq 1\}$ and $K_1^0 = \{v = (v_1, \ldots, v_n)^\top \in D : v_i \in \mathbb{Z}\ \forall i = 1, \ldots, n\}$. The set K_1^0 is the set of vertices of K_1. For $i = 1, \ldots, n$ define $q(i)$ to be the ith column of the $n \times n$-matrix

$$\begin{pmatrix} -1 & 0 & & & 0 & 0 \\ +1 & -1 & & & \vdots & \vdots \\ 0 & +1 & \ddots & & \vdots & \vdots \\ \vdots & 0 & & \ddots & 0 & 0 \\ \vdots & \vdots & & & -1 & 0 \\ 0 & 0 & & & +1 & -1 \end{pmatrix} \tag{4.10}$$

Let $\gamma = (\gamma^1, \ldots, \gamma^n)$ be a permutation of $(1, \ldots, n)$ and let $v^0, \ldots, v^n$ be elements of K_1^0 such that

$$v^{i+1} = v^i + q(i+1) \tag{4.11}$$

for $i = 0, \ldots, n$. Each such $\tau = \{v^0, \ldots, v^{n+1}\}$ is defined to be an n-simplex τ of K_1; each such τ is uniquely described by the pair (v^0, γ), so we say that

these correspond and write $\tau \approx (v^0, \gamma)$. Observe that

$$v^0 \succ v^1 \succ \ldots \succ v^n$$

is a lexicographically decreasing sequence. By adjoining the faces of the n-simplices the specification of K_1 is complete. The set $|K_1|$ is a triangulation of D under the correspondence $\alpha \mapsto \sum_{v \in K_1^0} \alpha(v)v$. Now define

$$\begin{aligned} h : |K_1| &\to S \times [1, \infty), \\ \alpha &\mapsto (x/\|x\|_1, \|x\|_1), \end{aligned} \tag{4.12}$$

where $x = \sum_{v \in K_1^0} \alpha(v)v$ and $\|x\|_1$ is of course the 1-norm of x.

The algorithm starts with $v^0 = (1, 0, \ldots, 0, 1)^\top$ and $\gamma = (n, 1, \ldots, n-1)$. Let $\tau \approx (v^1, \gamma)$ be an n-simplex of K_1. Let v^i, as in (4.11), be a vertex to be "dropped" as in the step of the algorithm. In our particular case exactly two vertices share the same label and this is one of them.

Table 4.1 shows how to select a new vertex to generate the n-simplex $\tau' \neq \tau$ of K_1 containing $\tau \setminus \{v^i\}$. In Figure 4.4 the complex K_1 is drawn for the case $n = 2$ and a possible beginning of the sequence of 2- and 1-simplices is also indicated.

	v^0 becomes	γ becomes
$i = 0$	$v^0 + q(\gamma^1)$	$(\gamma^2, \ldots, \gamma^n, \gamma^1)$
$1 \leq i \leq n-1$	v^0	$(\gamma^1, \ldots, \gamma^{i+1}, \gamma^i, \ldots, \gamma^n)$
$i = n$	$v^0 - q(\gamma^n)$	$(\gamma^n, \gamma^1, \ldots, \gamma^{n-1})$

Table 4.1.: Generating the n-simplex $\tau' \neq \tau$ of K_1 containing $\tau \setminus \{v^i\}$.

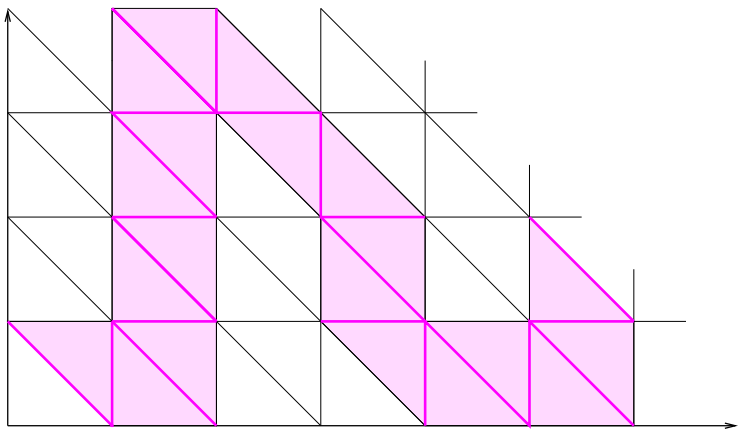


Figure 4.4.: The complex K_1 in the case $n = 2$.

Pseudo code

Now we put the theoretical development of Eaves and our adaptation to it into Algorithm 2. If the no joint increase condition (4.7) holds on a simplex S_r for $r > 0$, then this algorithm finds a point $x \in \Psi \cap S_r$. In favor of a brighter exposition we omitted possible performance improvements.

Algorithm 2 Finding $x \in \Psi \cap S_r$, for $r > 0$.

Require: $r > 0$ /* radius for S_r to find point in */
Require: $M > 1$ /* limit the number of loops by a bound on the maximal refinement */
Require: $\Gamma_\mu(s) \not\geq s \; \forall s \in S_r$
/* Initialization */
$v^0 \leftarrow (1, 0, \ldots, 0, 1)^\top$
$\gamma \leftarrow (n, 1, \ldots, n-1)$
let $v^{i+1} \leftarrow v^i + q(i+1)$, for $i = 1, \ldots, n$ /* q is defined in (4.10) */
let $\tau \approx (v^0, \gamma)$
repeat
/* FIRST STEP: find vertex to drop */
calculate labels $l(r \cdot h_2(|v^i|))$, for $i = 1, \ldots, n$ /* according to label function l defined in (4.8) */
let v^{i^*} be the vertex whose label appears twice and that is not equal to v^0
/* SECOND STEP: find new vertex, i.e., construct new simplex */
if $i^* = 0$ **then**
$v^0 \leftarrow v^0 + q(\gamma^1)$ and $\gamma \leftarrow (\gamma^2, \ldots, \gamma^n, \gamma^1)$
else if $1 \leq i^* \leq n-1$ **then**
/* leave v^0 as it is */
$\gamma \leftarrow (\gamma^1, \ldots, \gamma^{i+1}, \gamma^i, \ldots, \gamma^n)$
else
/* $i^* = n$ */
$v^0 \leftarrow v^0 - q(\gamma^n)$ and $\gamma \leftarrow (\gamma^n, \gamma^1, \ldots, \gamma^{n-1})$
end if
let $v^{i+1} \leftarrow v^i + q(i+1)$, for $i = 1, \ldots, n$
let $\tau \approx (v^0, \gamma)$
/* Terminate if refinement is getting too fine or if we have found a point in the decay set Ψ: */
until $h_2(\tau) > M$ or $\Gamma_\mu(r \cdot h_1(v^i)) \leq r \cdot h_1(v^i)$ for some i

From Ψ to Ω

Now that we can compute a point x in $\Psi \cap S_r$, the next step is to transform this point into a point $\tilde{x}$ in $\Omega \cap S_r$. The following lemmas will be useful to accomplish that.

Lemma 4.1.7 *Let $\Gamma \in \mathcal{G}^{n\times n}$ have a zero diagonal, let $\mu \in \mathrm{MAF}_n^n$, and $r > 0$. Assume $\Gamma_\mu(s) \not\geq s$ for all $s \in S_r$. Then the sets Ω_i and Ψ_i are star shaped with respect to $r \cdot e^i$. In particular, for $x \in \Psi_i \cap S_r$ and $\lambda \in [0, 1)$ the convex combination $z = (1 - \lambda)r^i + \lambda \cdot x$ is in $\Omega_i \cap S_r$.*

Proof. Assume $\|x\|_1 = r$ and $\Gamma_\mu(x)_i < x_i$. Since Γ has a zero diagonal, $\Gamma_\mu(x)_i$ does not depend on x_i. Hence for any y such that $0 \leq y_j \leq x_j$ for $j \neq i$ and $y_i \geq x_i$ by monotonicity we have $\Gamma_\mu(y)_i \leq \Gamma_\mu(x)_i < x_i \leq y_i$. Hence $y \in \Omega_i$. Analogously if only $x \in \Psi_i$. Clearly y can be chosen such that in addition $y \in S_r$. In particular the convex combination z is of this form. □

Lemma 4.1.8 *Let $\Gamma \in \mathcal{G}^{n\times n}$ have a zero diagonal, let $\mu \in \mathrm{MAF}_n^n$, and $r > 0$. Assume $\Gamma_\mu(s) \not\geq s$ for all $s \in S_r$. Then for any $x \in \Psi \cap S_r$ there is a point $\tilde{x} \in \Omega \cap S_r$ arbitrarily close to x.*

Proof. By Proposition 1.5.3 we know that $\Omega \cap S_r \neq \emptyset$ and by the monotonicity properties of Γ_μ and continuity we even have $\overline{\Omega \cap S_r} = \Psi \cap S_r$. We always have that $I_\Omega \cup I_b = \{1, \ldots, n\}$ where

$$I_\Omega(x) = \{i : \Gamma_\mu(x)_i < x_i\} \text{ and}$$
$$I_b(x) = \{i : \Gamma_\mu(x)_i = x_i\}.$$

Note that the set I_Ω is nonempty by the no joint increase condition. If $I_b(x) = \emptyset$ then $x \in \Omega$ and we are done.

Assume $I_b(x) \neq \emptyset$. Now let $x^0 = x$. We proceed inductively. We construct a finite sequence $x^1, x^2, \ldots$ and show that $I_b(x^k) \supsetneq I_b(x^{k+1})$. After finitely many steps we must have $I_b(x^k) = \emptyset$ for some $k \geq 1$.

Let $k \geq 0$ and assume $I_b(x^k) \neq \emptyset$. Pick $i \in I_b(x^k)$. Then for $\lambda \in (0, 1]$ sufficiently small define $x^{k+1} = (1-\lambda)x^k + \lambda r e^i$. Since the sets Ω_j are all open, if $x^k \in \Omega_j$ for $j \in I_\Omega$ then also $x^{k+1} \in \Omega_j$. Since Ψ_i is star shaped and by the preceding lemma we now have $x^{k+1} \in \Omega_i$. This implies $I_b(x^k) \supsetneq I_b(x^{k+1})$. Clearly as a convex combination of two points in S_r the point x^{k+1} is also in S_r. □

We formulate the last lemma more explicitly in Algorithm 3.

Algorithm 3 From $x \in \Psi \cap S_r$ to $\tilde{x} \in \Omega \cap S_r$ for $r > 0$.

Require: $r > 0$, $x \in \Psi \cap S_r$
Require: $\Gamma_\mu(s) \not\geq s \ \forall s \in S_r$
  $x^0 \leftarrow x$ /* start with boundary point of $\Omega \cap S_r$ */
  $k \leftarrow 0$
  $\tilde{x} \leftarrow x^k$
  while $I_b(x^k) \neq \emptyset$ **do**
    $i \leftarrow \min I_b(x^k)$ /* take coordinate i with $\Gamma(x^k)_i = x_i^k$ */
    $\lambda \leftarrow 1$
    repeat
      $\lambda \leftarrow \lambda/2$ /* determine step size such that $\Gamma(x^{k+1})_i < x_i^{k+1}$ */
      $x^{k+1} \leftarrow (1-\lambda)x^k + \lambda r e^i$
    until $\pi_{I_\Omega} \circ \Gamma_\mu(x^{k+1}) \ll \pi_{I_\Omega}(x^{k+1})$ **and** $\pi_i \circ \Gamma_\mu(x^{k+1}) < x_i^{k+1}$
    $\tilde{x} \leftarrow x^k$
    $k \leftarrow k+1$
  end while /* we're done when all coordinates i satisfy $\Gamma(x)_i < x_i$ */

4.1.3. Constructing D on $[0, w']$ for $w' \in \Omega$

With the results of the last section we are prepared to implement another constructive proof of Chapter 2. In the proof of Theorem 2.6.2 it has been shown how the diagonal operator D that is so essential for the results in Chapter 3 actually can be constructed locally. Since that proof is already formulated in such an instructive manner, we omit to produce an redundant algorithm here.

4.1.4. Constructing σ on $[0, w']$ for $w' \in \Omega$

The construction of the IOpS Lyapunov function in Chapter 3 crucially depends on the construction of a parameterized path $\sigma \in \mathcal{K}_\infty^n$ with image in Ω, such that the inverses of the component functions are locally Lipschitz continuous. Although for the general case there were several cases to distinguish in Section 2.3, for a numerical implementations the only important case is the construction of this path locally, i.e., on a sets $[0, w]$ for some $w \in \Omega$. In the proof of Proposition 2.3.1 it has been described how this can be performed. Again we intentionally omit to reproduce the algorithm, because it would not add any further insights on how to implement the proof of Proposition 2.3.1.

4.2. Application to formation control

In this section we consider an example application of our general ISS small gain theorem in the context of formation control. We exemplarily calculate gains, check the small gain condition and perform a numerical simulation.

We consider the following problem: A group of n vehicles has to maintain a desired formation, i.e., fixed distances and angles between vehicles, while at the same time they have to follow a reference trajectory. We assume that for technical reasons we have to manage to achieve this goal by means of autonomous control. By that we mean that each vehicle is equipped with a local controller and as control inputs only local information shall be available, in our case the distances and angles to neighbors of a vehicle, but possibly not to every vehicle in the group.

In [TPK04] formation control using ISS estimates has been considered. There a translational and rotational velocity control has been designed that renders the errors of the separation distance and relative bearing between a leading and a following vehicle input-to-state stable with respect to the error in front of the leading vehicle. This leads to a cascade of error propagation. Since the appearance of [Son89] it is known hat cascades of ISS systems are again ISS.

So our treatment of this problem enhances the results in [TPK04] in that we also allow "looking back" control. We use the same model, but instead of considering cascades only, we will also consider interconnections containing loops. We consider a formation of vehicles as depicted in Figure 4.5. The

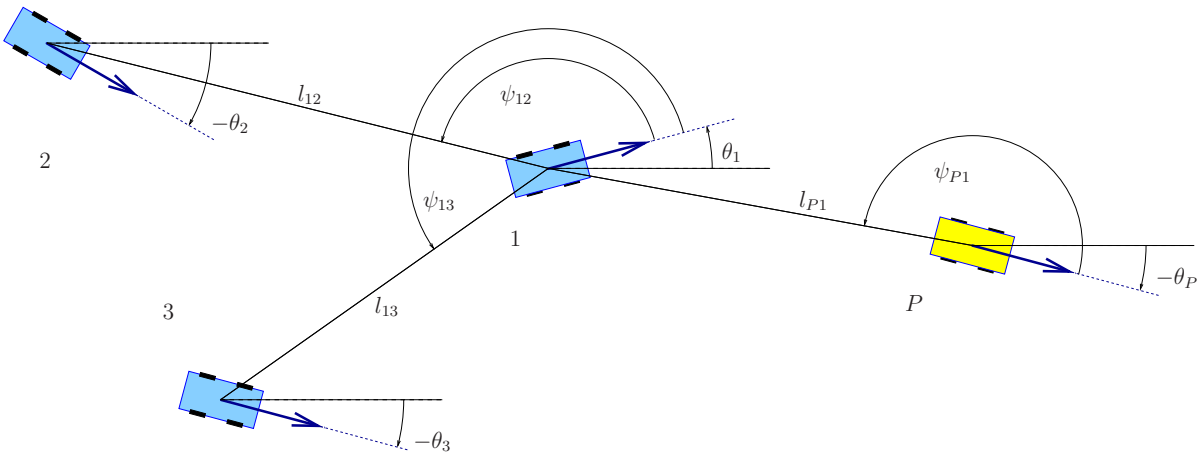


Figure 4.5.: A formation of three vehicles (1–3) following a phantom leader (P).

kinematic model of each vehicle is

$$\dot{x}_i = v_i \cos\theta_i, \quad \dot{y}_i = v_i \sin\theta_i, \quad \dot{\theta}_i = \omega_i$$

where (x_i, y_i) is the absolute position and θ_i the orientation of the vehicle. The translational and rotational velocities are v_i and ω_i, respectively. The relationship between leader i and follower j is expressed in terms of the separation distance l_{ij} and relative bearing ψ_{ij}. We consider the deviation of these variables from specification parameters l_{ij}^d, ψ_{ij}^d, i.e., our states are

$$\tilde{l}_{ij} = l_{ij}^d - l_{ij}, \quad \text{and} \quad \tilde{\psi}_{ij} = \psi_{ij}^d - \psi_{ij},$$

and we denote $\tilde{z}_{ij} = (\tilde{l}_{ij}, \tilde{\psi}_{ij})^\top$.

Let $\phi_{ij} = \theta_i - \theta_j$ and denote by d the distance from the wheel axis to a reference point. Then the leader-follower dynamics takes the form

$$\begin{pmatrix} \dot{\tilde{l}}_{ij} \\ \dot{\tilde{\psi}}_{ij} \\ \dot{\phi}_{ij} \end{pmatrix} = \begin{pmatrix} \cos\psi_{ij} & 0 \\ \frac{-\sin\psi_{ij}}{l_{ij}} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} - \begin{pmatrix} \cos(\phi_{ij}+\psi_{ij}) & d\sin(\phi_{ij}+\psi_{ij}) \\ \frac{-\sin(\phi_{ij}+\psi_{ij})}{l_{ij}} & \frac{d\cos(\phi_{ij}+\psi_{ij})}{l_{ij}} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_j \\ \omega_j \end{pmatrix}.$$

Now denote

$$A_{ij} = \begin{pmatrix} \cos\psi_{ij} & 0 \\ \frac{-\sin\psi_{ij}}{l_{ij}} & 1 \end{pmatrix}$$

and

$$B_{ij} = \begin{pmatrix} \cos(\phi_{ij}+\psi_{ij}) & d\sin(\phi_{ij}+\psi_{ij}) \\ \frac{-\sin(\phi_{ij}+\psi_{ij})}{l_{ij}} & \frac{d\cos(\phi_{ij}+\psi_{ij})}{l_{ij}} \end{pmatrix}.$$

In [TPK04] the control input

$$\begin{pmatrix} v_j \\ \omega_j \end{pmatrix} = B_{ij}^{-1} K^j \begin{pmatrix} \tilde{l}_{ij} \\ \tilde{\psi}_{ij} \end{pmatrix}$$

has been found using input-output feedback linearization, with a matrix $K^j = \mathrm{diag}(k_1^j, k_2^j)$ with $k_1^j, k_2^j > 0$, and the dynamics of ϕ_{ij} is already known to be stable, cf. [TPK04]. It has been shown that this control renders the error dynamics

$$\dot{\tilde{z}}_{ij} = A_{ij} \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} - B_{ij} \begin{pmatrix} v_j \\ \omega_j \end{pmatrix}$$

input to state stable from $\tilde{z}_{ki}$ to $\tilde{z}_{ij}$, where k is the number of the vehicle in front of vehicle i. We extend the velocity control for the first vehicle in our

group in Figure 4.5 to obtain

$$\begin{aligned}
\begin{pmatrix} v_1 \\ \omega_1 \end{pmatrix} &= B_{P1}^{-1} K^{1f} \tilde{z}_{P1} + B_{12}^{-1} K^{1bl} \tilde{z}_{12} + B_{13}^{-1} K^{1br} \tilde{z}_{13}, \\
\begin{pmatrix} v_2 \\ \omega_2 \end{pmatrix} &= B_{12}^{-1} K^{2f} \tilde{z}_{12}, \\
\begin{pmatrix} v_3 \\ \omega_3 \end{pmatrix} &= B_{13}^{-1} K^{2f} \tilde{z}_{13}.
\end{aligned}$$

Here K^f, K^{bl}, K^{br} denote the magnitude of the feedback from the front, behind left, and behind right, respectively. Each such K is of the form

$$K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

for some $k_1, k_2 > 0$. For simplicity we take $k_1 = k_2$.

With the above feedback control the formation dynamics takes the form

$$\dot{\tilde{z}}_{P1} = A_{P1} \begin{pmatrix} v_P \\ \omega_P \end{pmatrix} - K^{1f} \tilde{z}_{P1} - B_{P1}(B_{12}^{-1} K^{1bl} \tilde{z}_{12} + B_{13}^{-1} K^{1br} \tilde{z}_{13}) \tag{4.13}$$

$$\dot{\tilde{z}}_{12} = A_{12}(B_{P1}^{-1} K^{1f} \tilde{z}_{P1} + B_{12}^{-1} K^{1bl} \tilde{z}_{12} + B_{13}^{-1} K^{1br} \tilde{z}_{13}) - K^{2f} \tilde{z}_{12} \tag{4.14}$$

$$\dot{\tilde{z}}_{13} = A_{13}(B_{P1}^{-1} K^{1f} \tilde{z}_{P1} + B_{12}^{-1} K^{1bl} \tilde{z}_{12} + B_{13}^{-1} K^{1br} \tilde{z}_{13}) - K^{2f} \tilde{z}_{13}. \tag{4.15}$$

Auxiliary calculations Before we continue with stability estimates, we perform a few auxiliary calculations that are needed when we calculate ISS gains.

Each matrix B_{ij} is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/l_{ij} \end{pmatrix} \cdot R \cdot \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix},$$

where R is a rotation matrix. The inverse of a rotation matrix is again a rotation matrix, so the inverse of B_{ij} is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^{-1} \cdot R^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1/l_{ij} \end{pmatrix}^{-1}.$$

Therefore we can estimate the matrix norm of the matrix $B_{ij} B_{lk}$ as

$$\|B_{ij} B_{lk}^{-1}\|_2 = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1/l_{ij} \end{pmatrix} \cdot \tilde{R} \cdot \begin{pmatrix} 1 & 0 \\ 0 & l_{lk} \end{pmatrix} \right\|_2 \leq \max\{1, 1/l_{ij}, l_{lk}, l_{lk}/l_{ij}\} \tag{4.16}$$

where $\tilde{R}$ is again some rotation matrix.

The operator norm of A_{ij} can be estimated by

$$\|A_{ij}\|_2 \leq \sqrt{1 + 1/d^2} =: C_A \tag{4.17}$$

provided that $l_{ij} > d$ can be maintained (e.g., to avoid collisions).

The norm of B_{ij}^{-1} can be estimates as

$$\|B_{ij}^{-1}\|_2 \leq \max\{1, l_{ij}/d, l_{ij}, 1/d\} = \max\{l_{ij}/d, l_{ij}, 1/d\}.$$

Using the fact that

$$l_{ij} = l_{ij}^d - \tilde{l}_{ij} \leq l_{ij}^d + \|\tilde{z}_{ij}\|, \tag{4.18}$$

and assuming in addition $l_{ij} \leq l^{\mathrm{MAX}}$ for some l^{MAX} for all times, we also find the estimates

$$\|B_{ij}^{-1}\|_2 \leq \max\{l^{\mathrm{MAX}}/d, l^{\mathrm{MAX}}, 1/d\} =: C_B \quad \text{and} \tag{4.19}$$

$$\|B_{ij}^{-1}\|_2 \leq \max\{(l_{ij}^d + \|\tilde{z}_{ij}\|)/d, l_{ij}^d + \|\tilde{z}_{ij}\|, 1/d\} =: g_{ij}(\|\tilde{z}_{ij}\|). \tag{4.20}$$

We will use that estimate that is more appropriate or less conservative under the circumstances.

With (4.18) and $l > d$ we can in turn refine (4.16) to

$$\|B_{ij}B_{lk}^{-1}\|_2 \leq \max\{1/d, l_{lk}^d + \|\tilde{z}_{lk}\|, (l_{lk}^d + \|\tilde{z}_{lk}\|)/d\} = g_{lk}(\|\tilde{z}_{lk}\|). \tag{4.21}$$

Stability considerations Next we use the Lyapunov function candidates

$$V_j(\tilde{z}_{ij}) = \frac{1}{2}\tilde{z}_{ij}^\top (K^{jf})^{-1}\tilde{z}_{ij}$$

to get local ISS Lyapunov estimates of the form

$$\nabla V_j(\tilde{z}_{ij})\dot{\tilde{z}}_{ij} \leq -\alpha_{j3}(\|\tilde{z}_{ij}\|) \quad \text{if} \quad \|\tilde{z}_{ij}\| \geq \sum_{kl} \gamma_{ij,kl}(\tilde{z}_{kl}) \tag{4.22}$$

for suitable $\mathcal{K}_\infty$ functions $\alpha_{j3}, \gamma_{ij,kl}$. Since $k^{jf} > 0$ for $j = 1, 2, 3$, there clearly exist $\mathcal{K}_\infty$ functions α_{j1}, α_{j2} such that

$$\alpha_{j1}(\|\tilde{z}_{ij}\|) \leq V_j(\tilde{z}_{ij}) \leq \alpha_{j2}(\|\tilde{z}_{ij}\|)$$

for $j = 1, 2, 3$.

By the same argument already used in [Son89] it is evident that (local) ISS Lyapunov gains are also valid (local) ISS gains for trajectory estimates. So once we have found the gains $\gamma_{ij,kl}$ in (4.22) we can apply our local ISS small-gain Theorem 3.3.11 to judge, whether for a given choice of system parameters $l, d, k^{1f}, \ldots$ the error dynamics (4.13)–(4.15) is locally ISS from $(v_P, \omega_P)^\top$ to $\tilde{z} = (\tilde{z}_{P1}^\top, \tilde{z}_{12}^\top, \tilde{z}_{13}^\top)^\top$.

Gain calculation Fix some $\varepsilon \in (0,1)$. Using (4.13) we obtain

$$\begin{aligned}\nabla V_1(\tilde{z}_{P1})\dot{\tilde{z}}_{P1} \leq & \|\tilde{z}_{P1}\| \frac{1}{k^{1f}} \|A_{P1}\| \left\| \begin{pmatrix} v_p \\ \omega_P \end{pmatrix} \right\| - \varepsilon \|\tilde{z}_{P1}\|^2 - (1-\varepsilon)\|\tilde{z}_{P1}\|^2 \\ & + \|\tilde{z}_{P1}\| \frac{k^{1bl}}{k^{1f}} \|B_{P1}B_{12}^{-1}\| \|\tilde{z}_{12}\| + \|\tilde{z}_{P1}\| \frac{k^{1br}}{k^{1f}} \|B_{P1}B_{13}^{-1}\| \|\tilde{z}_{13}\| \\ \leq & -\varepsilon \|\tilde{z}_{P1}\|^2 \end{aligned}$$

provided that

$$\begin{aligned}\|\tilde{z}_{P1}\| \geq \frac{1}{1-\varepsilon} \Bigg[& \frac{\|A_{P1}\|}{k^{1f}} \left\| \begin{pmatrix} v_p \\ \omega_P \end{pmatrix} \right\| \\ & + \frac{k^{1bl}}{k^{1f}} \|B_{P1}B_{12}^{-1}\| \|\tilde{z}_{12}\| + \frac{k^{1br}}{k^{1f}} \|B_{P1}B_{13}^{-1}\| \|\tilde{z}_{13}\| \Bigg]. \end{aligned}$$

Using (4.21) and (4.17) we may choose

$$\begin{aligned} \gamma_{P1,P}\left(\left\| \begin{pmatrix} v_p \\ \omega_P \end{pmatrix} \right\| \right) &= \frac{1}{1-\varepsilon} \frac{C_A}{k^{1f}} \left\| \begin{pmatrix} v_p \\ \omega_P \end{pmatrix} \right\| \\ \gamma_{P1,12}(\|\tilde{z}_{12}\|) &= \frac{1}{1-\varepsilon} \frac{k^{1bl}}{k^{1f}} g_{12}(\|\tilde{z}_{12}\|)\|\tilde{z}_{12}\| \\ \gamma_{P1,13}(\|\tilde{z}_{13}\|) &= \frac{1}{1-\varepsilon} \frac{k^{1br}}{k^{1f}} g_{13}(\|\tilde{z}_{13}\|)\|\tilde{z}_{13}\| \end{aligned}$$

Similarly, using in addition (4.19), we compute

$$\begin{aligned}\nabla V_2(\tilde{z}_{12})\dot{\tilde{z}}_{12} \leq & -\|\tilde{z}_{12}\|^2 + \frac{k^{1bl}}{k^{2f}} \|A_{12}\| \|B_{12}^{-1}\| \|\tilde{z}_{12}\|^2 \\ & + \|\tilde{z}_{12}\| \frac{\|A_{12}\|}{k^{2f}} (k^{1f}\|B_{P1}^{-1}\| \|\tilde{z}_{P1}\| + k^{1br}\|B_{13}^{-1}\| \|\tilde{z}_{13}\|) \\ \leq & -\|\tilde{z}_{12}\|^2 + \underbrace{\frac{k^{1bl}}{k^{2f}} C_A C_B}_{=:C_{12}} \|\tilde{z}_{12}\|^2 \\ & + \|\tilde{z}_{12}\| \frac{C_A}{k^{2f}} (k^{1f} g_{P1}(\|\tilde{z}_{P1}\|)\|\tilde{z}_{P1}\| + k^{1br} g_{13}(\|\tilde{z}_{13}\|)\|\tilde{z}_{13}\|). \end{aligned} \tag{4.23}$$

Now suppose that $C_{12} \in (0,1)$, so (4.23) becomes

$$\begin{aligned}\nabla V_2(\tilde{z}_{12})\dot{\tilde{z}}_{12} \leq & -(1-C_{12})\|\tilde{z}_{12}\|^2 \\ & + \|\tilde{z}_{12}\|\frac{C_A}{k^{2f}}(k^{1f}g_{P1}(\|\tilde{z}_{P1}\|)\|\tilde{z}_{P1}\| + k^{1br}g_{13}(\|\tilde{z}_{13}\|)\|\tilde{z}_{13}\|) \\ \leq & -\varepsilon(1-C_{12})\|\tilde{z}_{12}\|^2\end{aligned}$$

if

$$\begin{aligned}(1-\varepsilon)(1-C_{12})\|\tilde{z}_{12}\|^2 \geq \|\tilde{z}_{12}\|\frac{C_A}{k^{2f}}(k^{1f}g_{P1}(\|\tilde{z}_{P1}\|)\|\tilde{z}_{P1}\| \\ + k^{1br}g_{13}(\|\tilde{z}_{13}\|)\|\tilde{z}_{13}\|).\end{aligned}$$

From here we obtain the gains from the $\tilde{z}_{P1}$ and $\tilde{z}_{13}$ systems to the $\tilde{z}_{12}$ system as

$$\begin{aligned}\gamma_{12,P1}(\|\tilde{z}_{P1}\|) &= \frac{C_A}{(1-\varepsilon)(1-C_{12})}\frac{k^{1f}}{k^{2f}}g_{P1}(\|\tilde{z}_{P1}\|)\|\tilde{z}_{P1}\| \\ \gamma_{12,13}(\|\tilde{z}_{13}\|) &= \frac{C_A}{(1-\varepsilon)(1-C_{12})}\frac{k^{1br}}{k^{2f}}g_{13}(\|\tilde{z}_{13}\|)\|\tilde{z}_{13}\|.\end{aligned}$$

Analogously, defining $C_{13} := \frac{k^{1br}}{k^{3f}}C_AC_B$ and assuming $C_{13} \in (0,1)$, we obtain

$$\begin{aligned}\gamma_{13,P1}(\|\tilde{z}_{P1}\|) &= \frac{C_A}{(1-\varepsilon)(1-C_{13})}\frac{k^{1f}}{k^{3f}}g_{P1}(\|\tilde{z}_{P1}\|)\|\tilde{z}_{P1}\| \\ \gamma_{13,12}(\|\tilde{z}_{12}\|) &= \frac{C_A}{(1-\varepsilon)(1-C_{13})}\frac{k^{1bl}}{k^{3f}}g_{12}(\|\tilde{z}_{12}\|)\|\tilde{z}_{12}\|\end{aligned}$$

as the gains from the $\tilde{z}_{P1}$ and $\tilde{z}_{12}$ systems to the $\tilde{z}_{13}$ system.

Formation stability The gain matrix takes the form

$$\Gamma = \begin{pmatrix} 0 & \gamma_{P1,12} & \gamma_{P1,13} \\ \gamma_{12,P1} & 0 & \gamma_{12,13} \\ \gamma_{13,P1} & \gamma_{13,12} & 0 \end{pmatrix}$$

and we have $\mu = \Sigma$. As mentioned above, although we used Lyapunov functions to calculate the gains, these gains are in fact valid for trajectory estimates as well. We now have to choose the system parameters very carefully, so that our local small-gain Theorem 3.3.11 applies.

We can formulate the main result of this section as follows:

Theorem 4.2.1 *For the formation of three vehicles the error dynamics given by* (4.13)–(4.15) *is locally input-to-state stable (LISS) from* $(v_P, \omega_P)^\top$ *to* $\tilde{z} = (\tilde{z}_{P1}^\top, \tilde{z}_{12}^\top, \tilde{z}_{13}^\top)^\top$ *for suitable choices of parameters.*

'Numerical' proof. Using a MATLAB implementation of the algorithms in Section 4.1 we find a vector

$$w = \begin{pmatrix} 3.2113 \\ 2.4606 \\ 2.5555 \end{pmatrix}$$

satisfying

$$\begin{pmatrix} 3.2106 \\ 2.4601 \\ 2.5550 \end{pmatrix} \approx \Gamma_\Sigma(w) \ll w$$

for the following choice of parameters:

$$\begin{aligned} l_{P1}^d &= 3 & l_{12}^d &= 5 & l_{13}^d &= 5 \\ \Psi_{P1}^d &= \pi & \Psi_{12}^d &= \arccos(-3/5) & \Psi_{13}^d &= 2\pi - \arccos(-4/5) \\ k^{1f} &= 165 & k^{2f} &= 23000 & k^{3f} &= 22000 \\ k^{1bl} &= 3 & k^{1br} &= 4 & & \\ d &= 0.25 & l^{\mathrm{MAX}} &= 8 & \varepsilon &= 0.001. \end{aligned}$$

The constants C_A, C_B, C_{13}, and C_{13} are then given by

$$\begin{aligned} C_A &= 4.1231 & C_B &= 32 \\ C_{12} &= 0.0172 & C_{12} &= 0.0240, \end{aligned}$$

so the assumptions we made during the calculation of the gains are all satisfied. Using that MATLAB implementation again, we validate that $\Gamma_\Sigma^k(w) \to 0$ as $k \to \infty$. Hence the local small-gain condition (3.50) is satisfied.

Now from the local ISS small-gain Theorem 3.3.11 we conclude that the error dynamics is indeed LISS as claimed. □

Simulation For the same set of parameters as in the preceding paragraph we confirm the statement of Theorem 4.2.1 by integrating system (4.13)–(4.15). Figure 4.6 exemplarily shows error plots as well as the resulting vehicle trajectories for a sine shaped input signal. The figures show that after a transient overshoot the norm of the signals very rapidly becomes bounded by an asymptotic estimate, indicating the validity of our LISS estimates, at least for small inputs.

However, for larger input disturbances LISS can not be concluded anymore and the simulations also indicate that the above feedback control might not be the most suitable way to control vehicle formations. As the controller parameters have to be chosen such that in particular

$$k^{2f} \gg k^{1f} \gg k^{1bl},$$

it is clear that vehicles at the rear of a formation, especially if we considered even larger formations, would react very sensitive to disturbances of the input signal fed to the leading vehicle. This leads to rampant movements of the trailing vehicles and is of course undesirable. Hence the stability conclusion given by Theorem 4.2.1 not only from the theoretical perspective of a local nature, but also in practice. It remains unclear whether other control techniques lead to more practical stability estimates, and autonomous control of vehicle formations with limited communication is still an active field of research.

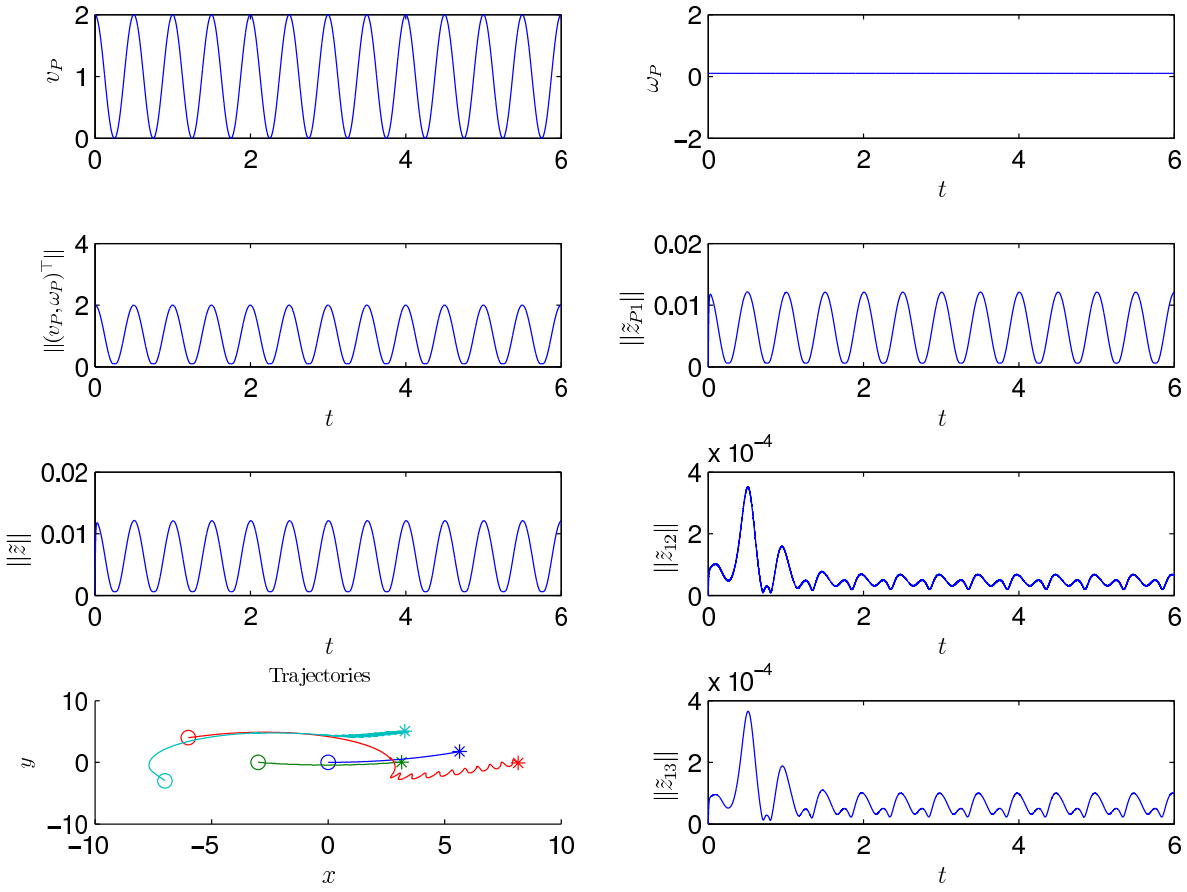


Figure 4.6.: Inputs and error plots for the formation dynamics (4.13)–(4.15). Here $\tilde{z} := (\tilde{z}_{P1}^\top, \tilde{z}_{12}^\top, \tilde{z}_{13}^\top)^\top$. The vehicles start at the positions denoted by little circles, the asterisks indicate their end positions. The initial formation has the form as in Figure 4.5. Vehicles 2 and 3 actually move backwards after about one time unit.

A. Nonsmooth analysis

Here we recall some facts from nonsmooth analysis: This theory deals with calculus of function that are only locally Lipschitz continuous instead of being differentiable. Nevertheless, concepts very similar to those from the familiar "smooth" calculus are at hand. This excerpt is based on the book [CLSW98].

In the following section we recall a few definitions and quote some results about generalized gradients.

Let X be a real Banach space. For $x \in X$ and $\varepsilon > 0$ by $B(x, \varepsilon)$ we denote the set $\{y \in X : \|x - y\| < \varepsilon\}$. On the dual space X^* of X we use the norm

$$\|\zeta\|_* := \sup\{\langle \zeta, v\rangle : v \in V, \|v\| = 1\},$$

where $\langle \zeta, v\rangle$ is the pairing of ζ and v.

A function $f : X \to \mathbb{R}$ is *Lipschitz of rank* K near $x \in X$, if there exists an $\varepsilon > 0$ and a $K \geq 0$, such that for all $y, z \in B(x, \varepsilon)$ we have

$$|f(y) - f(z)| \leq K\|y - z\|.$$

The function f is *locally Lipschitz*, if for every $x \in X$ it is Lipschitz of some rank $K = K(x)$. The *generalized directional derivative* of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{f(y + tv) - f(y)}{t}$$

A multivalued function $F : X \to 2^X$ is *upper semicontinuous* at x, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x - y\| < \delta \implies F(y) \subset F(x) + B(0, \varepsilon).$$

A function $g : X \to \mathbb{R}$ is *positively homogeneous* if $g(\lambda x) = \lambda g(x)$ for all $\lambda \geq 0$ and all $x \in X$, and *subadditive* if $g(v + w) \leq g(v) + g(w)$ for all $v, w \in X$.

Proposition A.1 *Let $f : X \to \mathbb{R}$ be Lipschitz of rank K near $x \in X$. Then*

1. *the function $v \mapsto f^\circ(x; v)$ is finite, positively homogeneous, subadditive on X, and satisfies*
$$|f^\circ(x; v)| \leq K\|v\|;$$

2. *$f^\circ(x;v)$ is upper semicontinuous as a function of (x,v) and, as a function of v alone, it is Lipschitz of rank K on X;*

3. *$f^\circ(x;-v) = (-f)^\circ(x;v)$.*

For a proof see [CLSW98, Prop. 2.1.1, p.70]. We define the *generalized gradient* ∂f of the function $f : X \to \mathbb{R}$ by

$$\partial f(x) := \{\zeta \in X^* : f^\circ(x;v) \geq \langle \zeta, v\rangle \ \forall v \in X\}.$$

We state a couple of properties of generalized gradients. The first one is stated as an exercise in [CLSW98, p.73]. For the convenience of the reader we give a proof.

Proposition A.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$. Then $f^\circ(x;v) = \langle f'(x), v\rangle$ and $\partial f(x) = \{f'(x)\}$.*

Proof. Since f is continuously differentiable, we have $f^\circ(x;v) = Df(x;v)$ the directional derivative of f. But $Df(x;v)$ is just equal to $\langle f'(x), v\rangle$.

We have $f'(x) = \zeta$ if and only if $\langle f'(x), v\rangle = \langle \zeta, v\rangle$ for all $v \in \mathbb{R}^n$, which is equivalent to $f^\circ(x;v) \geq \langle \zeta, v\rangle$ for all $v \in \mathbb{R}^n$, and this in turn to $\zeta \in \partial f(x)$. □

The following result is also taken from [CLSW98].

Proposition A.3 *Let $f : X \to \mathbb{R}$ be Lipschitz of rank K near $x \in X$. Then*

1. *$\partial f(x)$ is a nonempty, convex, weak*-compact subset of X^*, and $\|\zeta\|_* < K$ for every $\zeta \in \partial f(x)$;*

2. *for every $v \in X$ we have*

$$f^\circ(x;v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial f(x)\};$$

3. *$\zeta \in \partial f(x)$ if and only if $f^\circ(x;v) \geq \langle \zeta, v\rangle$ for all $v \in X$;*

4. *if $\{x_i\}$ and $\{\zeta_i\}$ are sequences in X and X^* such that $\zeta_i \in \partial f(x_i)$ for each i, and if x_i converges to x and ζ_i is a weak* cluster point of the sequence $\{\zeta_i\}$, then we have $\zeta \in \partial f(x)$;*

5. *if X is finite dimensional, then ∂f is upper semicontinuous at x.*

This result has been proved in [CLSW98, Proposition 2.1.5, p.73]. Next we gather facts relating nonsmooth calculus to standard calculus:

Theorem A.4 *1. Let $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, be Lipschitz near $x \in X$ and let λ_i be scalars, $i = 1, \ldots, n$. Then $f := \sum_i \lambda_i f_i$ is Lipschitz near x and*

$$\partial \left(\sum_i \lambda_i f_i \right)(x) \subset \sum_i \lambda_i \partial f_i(x).$$

2. ***(Lebourg's Mean Value Theorem)*** *Let $x, y \in X$ and let $f : X \to \mathbb{R}$ be Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point u in (x, y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle.$$

3. ***(The Chain Rule)*** *Let $F : X \to \mathbb{R}^n$ be Lipschitz near x, and let $g : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near $F(x)$. Then the function $f(x') := g(F(x'))$ is Lipschitz near x, and we have*

$$\partial f(x) \subset \overline{co}^* \{ \partial \langle \gamma, F(\cdot) \rangle(x) : \ \gamma \in \partial g(F(x)) \},$$

where $\overline{co}^*$ denotes the w^*-closed convex hull.

Proof. From [CLSW98, Proposition 2.2.1, p. 75] we have for scalars λ and functions f that Lipschitz near x:

$$\partial(\lambda f)(x) = \lambda \partial f(x).$$

Also on [CLSW98, p. 75] it has been proved that $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$. Both results put together give the first claim. The second and third claim are [CLSW98, Proposition 2.2.4, p. 75] and [CLSW98, Proposition 2.2.5, p. 76], respectively. □

The following result relates Lipschitz continuity to differentiability almost everywhere. A property is said to hold almost everywhere, if the set where it fails to hold has Lebesgue measure zero.

Theorem A.5 (Rademacher's Theorem) *Let $U \subset \mathbb{R}^n$ be open. Let $f : U \to \mathbb{R}$ be locally Lipschitz. Then f is differentiable almost everywhere on U.*

See, e.g., [Eva98, Theorem 5.8.6, p.281].

Corollary A.6 *Let $U \subset \mathbb{R}^n$ be open. Let $f : U \to \mathbb{R}^m$ be locally Lipschitz. Then f is differentiable almost everywhere on U.*

Proof. This is a direct consequence of Theorem A.5, by noting that a function $f : U \to \mathbb{R}^m$ is locally Lipschitz if and only if every component function $f_i : U \to \mathbb{R}$ is locally Lipschitz. □

Theorem A.7 (Generalized Gradient Formula) *Let $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz at x. Let $U \subset \mathbb{R}^n$ have Lebesgue measure zero. Denote by U_f the set of points in $\mathbb{R}^n$ where f fails to be differentiable. Then we have*

$$\partial f(x) = co\left\{\lim_{i\to\infty} \nabla f(x_i) : \ x_i \to x,\, x_i \notin U,\, x_i \notin U_f\right\}.$$

For a proof see [CLSW98, Theorem 2.8.1, p.93]. The next result is an immediate consequence of Theorem A.7, but has not been stated explicitly in [CLSW98].

Corollary A.8 (Chain Rule 2) *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz, denote by U_f and U_g the sets of points where f and, respectively, g fail to be differentiable. Suppose $g^{-1}[U_f]$ has zero Lebesgue measure. Then for all $x \in \mathbb{R}^m$ we have*

$$\partial(f \circ g)(x) = co\left\{\lim_{i\to\infty} \nabla f(g(x_i)) \cdot Dg(x_i) : \ x_i \to x,\, x_i \notin U_g,\, g(x_i) \notin U_f\right\}.$$

In other words

$$\partial(f \circ g)(x) = \nabla f(g(x)) \cdot Dg(x)$$

for almost every $x \in X$.

Example A.9 *Let $f, g \in \mathcal{K}_\infty$ be locally Lipschitz, denote by U_f and U_g the sets of points where f and, respectively, g fail to be differentiable. Note that also $g^{-1} \in \mathcal{K}_\infty$ and $(g^{-1})' = 1/(g' \circ g^{-1})$ exists almost everywhere. From Rademacher's Theorem we know that U_f and U_g have Lebesgue measure 0. Then the Lebesgue measure of $g^{-1}[U_f]$ is given by*

$$\begin{aligned}
\lambda(g^{-1}[U_f]) &= \int_{\mathbb{R}_+} 1_{g^{-1}[U_f]}(y) d\lambda(y) \\
&= \int_{\mathbb{R}_+} 1_{g^{-1}[U_f]}(g^{-1}(x)) \cdot \frac{1}{g' \circ g^{-1}(x)} d\lambda(x) \\
&= \int_{\mathbb{R}_+} 1_{U_f}(x) \frac{1}{g' \circ g^{-1}(x)} d\lambda(x) = \int_{U_f} \frac{1}{g' \circ g^{-1}(x)} d\lambda(x) = 0
\end{aligned}$$

because U_f is a null set. Hence $(f \circ g)' = (f' \circ g) \cdot g'$ almost everywhere.

Corollary A.8 together with the following result on the generalized gradient of the maximum of n functions will be an essential argument in the chapter on ISS Lyapunov functions. The next result is taken from [CLSW98, p.83].

Proposition A.10 *For $i = 1, \ldots, n$ let $f_i : X \to \mathbb{R}$ be Lipschitz near $x \in X$. Set $f(x) = max_{1\le i\le n} f_i(x)$. Then f is Lipschitz near x and*

$$\partial f(x) \subset co\left\{\bigcup_{i\in M(x)} \partial f_i(x)\right\},$$

where $M(x) = \{i \in \{1, \ldots, n\} : f_i(x) = f(x)\}$.

A survey on nonsmooth analysis in control theory can be found in [Cla01]. All definitions and results in this section are stated in [CLSW98], with the exception of Corollary A.6 and Corollary A.8, Example A.9. Rademacher's Theorem has been taken from [Eva98].

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